Characterizations of Non-simple Greedoids and Antimatroids based on Greedy Algorithms

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Abstract
A language containing no words with repeated elements is simple. And a language which is not necessary simple is called non-simple in this paper. Björner and Ziegler [3] presented three plausible extensions of greedoids to non-simple languages. We shall show that when choosing one of their definitions they called 'polygreedoid' and replacing generalized bottleneck functions of the objective function by an extended notion defined in this note, the algorithmic characterization of (simple) greedoids by Goecke, Korte and Lovász [7] can be naturally generalized to non-simple greedoids. Björner, Lovász and Shor [2] introduced non-simple antimatroids relating to chip-firing games on graphs. We also show that the algorithmic characterization of (simple) antimatroids established by Boyd and Faigle [4] can be similarly extended to non-simple antimatroids as well.

1 Introduction
The notion of greedoids was invented by Korte and Lovász around 1980, which offers a wide framework for the combinatorial structures related to exchange principles and greedy algorithms. It includes as a special case both matroids and antimatroids as well as various decomposition schemes such as Gaussian elimination of matrices, ear decomposition of 2-connected graphs, bisimplicial elimination in bipartite graphs, series-parallel reduction of graphs, retract elimination in directed graphs, etc.

The systematic studies of antimatroids seem to be started by Edelman [5] and Jamison [8]. (See also Edelman and Jamison [6],) They studied a convex geometry, which is the complement of an antimatroid, as an abstraction of convexity. Matroids and antimatroids are 'dual' or 'antipodal' to each other in the sense that the closure of a matroid has the Steinitz-MacLane exchange property, while that of an antimatroid satisfies the anti-exchange law. Antimatroids typically arise as a collection of allowed sequences of deletion or scanning on a combinatorial object such as poset shelling, node or edge shelling of trees, simplicial elimination sequences in triangulated graphs, vertex shelling of polytopes, node-search or line-search on graphs, and so on. For more details of greedoids and antimatroids, we refer to [10].
In the algorithmic aspect, a linear objective function can be optimized by a greedy algorithm on matroids, and conversely this property characterizes matroids. In contrast with this, optimizing a linear function on greedoids is an intractable problem.

Goecke, Korte and Lovász [7] have shown that on greedoids the maximization of a 'generalized bottleneck function' is achieved by a greedy procedure, and vice versa. Similarly, Boyd and Faigle [4] have shown that on an antimatroid, a 'nested bottleneck function' is optimized by a greedy algorithm, and that this conversely characterizes antimatroids. Boyd and Faigle's result is an extension of Lawler's one [11] that a single-machine scheduling problem obeying a precedence constraint of a partial order among jobs can be solved by a greedy procedure when the objective is a certain bottleneck function.

Originally greedoids and antimatroids are defined as simple languages, i.e. those which contain no words with repeated elements. Björner and Ziegler [3] studied three possible extensions of greedoids to non-simple languages. Björner, Lovász and Shor [2] introduced non-simple antimatroids in connection with chip-firing games of graphs. In this paper we shall show that the algorithmic characterization of (simple) greedoids by Goecke, Korte and Lovász [7] and that of (simple) antimatroids by Boyd and Faigle [4] can be naturally extended to the corresponding non-simple cases respectively.

2 Greedoids and antimatroids

Let $E$ denote a finite non-empty set throughout this note. A word on $E$ is a finite sequence of elements of $E$, and $\epsilon$ denotes a word of null length. For a word $\alpha = a_1 \cdots a_k$, $\tilde{\alpha}$ denotes the set of elements of $\alpha$, and $|\alpha|$ is its length. A set of words is called a language. For the sake of simplicity, we assume in this paper that languages are all finite. For a language $L$, $\rho(L)$ is the maximal length of words of $L$. If every maximal word has the same finite length, $L$ is called pure. A word is simple if it has no repeated elements, and a language is called simple if its words are all simple. A language is non-simple if it is not restricted to be simple words. That is, 'non-simple' implies 'not necessarily simple' throughout this paper. A language $L$ is called left-hereditary if $\epsilon \in L$ and a left-prefix of its word belongs to $L$ again, i.e. $\alpha \beta \in L$ implies $\alpha \in L$.

For a fixed number $r$, $\{\alpha \in L : |\alpha| \leq r\}$ is called a truncation of $L$. Let $L$ be a left-hereditary simple language on $E$. $L$ is called a greedoid if it satisfies

$$\text{(Gr)} \quad \alpha, \beta \in L, |\alpha| > |\beta| \implies \exists x : x \in \tilde{\alpha} \setminus \tilde{\beta}, \beta x \in L.$$

And $L$ is called an antimatroid if it satisfies

$$\text{(An)} \quad \alpha, \beta \in L, \tilde{\alpha} \nsubseteq \tilde{\beta} \implies \exists x : x \in \tilde{\alpha} \setminus \tilde{\beta}, \beta x \in L.$$

As is easily seen, a greedoid and an antimatroid are simple pure languages. Although a greedoid as well as an antimatroid can be formulated both as a language (ordered version) and as a family of sets (unordered version), we treat in this paper only with languages, i.e. the ordered version.

Let $L$ be a simple or non-simple language on $E$, and suppose further $L$ is left-hereditary and pure. Let $F$ be a real-valued function on $L$. Then associated with $F$, we can define a function

$$w(\alpha) = \min\{F(a_1 a_2 \cdots a_i) : i = 1, \ldots, k\} \quad \text{for} \quad \alpha = a_1 \cdots a_k \in L. \quad (2.1)$$
And we shall consider the optimization problem below:

$$\max \ w(\alpha)$$

subject to \( \alpha \in L, \ |\alpha| = \rho(L) \) \hspace{1cm} (2.2)

The following procedure gives a candidate of solution of (2.2).

< GREEDY >

\[ \alpha \leftarrow \epsilon ; \]

\textbf{while} \( \Gamma(\alpha) \neq \emptyset \) and \( |\alpha| < \rho(L) \) \textbf{do} begin

\[ \text{choose } x \in \Gamma(\alpha) \text{ such that } w(\alpha x) \text{ is maximum} ; \]

\[ \alpha \leftarrow \alpha x ; \]

\textbf{end}

where \( \Gamma(\alpha) \) is the set of elements which succeed \( \alpha \), i.e. \( \Gamma(\alpha) = \{ x \in E : \alpha x \in L \} \).

We shall first describe the algorithmic characterization of greedoids. Let \( Z_+ \) be the set of nonnegative integers, and \( f \) a real-valued function on \( E \times Z_+ \) such that it is monotone with respect to the second variable, i.e.

\[ i \leq j \implies f(e, i) \leq f(e, j) \] \hspace{1cm} (2.3)

Setting \( F(a_1 \cdots a_i) = f(a_i, i) \) in (2.1) gives rise to a bottleneck function such that

\[ w(\alpha) = \min_{i=1, \ldots, k} \{ f(a_i, \{a_1, a_2, \ldots, a_i\}) \} \hspace{1cm} (\alpha = a_1 \cdots a_k \in L) \] \hspace{1cm} (2.4)

which is called a generalized bottleneck function. Then we have

\textbf{Theorem 2.1 (Goecke, Korte and Lovász [7])} If \( L \) is a greedoid, then GREEDY gives an optimal solution for a generalized bottleneck function. Conversely, for a left-hereditary simple pure language \( L \), if the solution given by GREEDY is always optimal for any generalized bottleneck function, then \( L \) is a greedoid. \( \square \)

We have a similar algorithmic characterization for antimatroids as well, the result of which has the origin in the work of Lawler [11] on a single-machine scheduling problem. He showed that when a precedence constraint is given as a partial order among jobs and the objective function to be minimized is a certain bottleneck function, a single-machine scheduling problem can be solved by a greedy procedure. Extending this, Boyd and Faigle [4] have shown that an analogous result holds for antimatroids and in this case the converse is also true. Let \( f \) be a real-valued function on \( E \times 2^E \). And suppose \( f \) to be monotone with respect to the second variable, i.e.

\[ A \subseteq B \implies f(e, A) \leq f(e, B) \] \hspace{1cm} (2.5)

Setting \( F(a_1 \cdots a_i) = f(a_i, \{a_1, \ldots, a_i\}) \) in (2.1), we have a function on \( L \) such that

\[ w(\alpha) = \min_{i=1, \ldots, k} \{ f(a_i, \{a_1, a_2, \ldots, a_i\}) \} \hspace{1cm} (\alpha = a_1 \cdots a_k \in L) \] \hspace{1cm} (2.6)

which is called a nested bottleneck function. Then,
Theorem 2.2 (Boyd and Faigle [4]) If \( L \) is a truncation of an antimatroid, the algorithm GREEDY gives an optimal solution for a nested bottleneck function. Conversely, for a left-hereditary simple pure language \( L \), if the solution given by GREEDY is always optimal for any nested bottleneck function, then \( L \) is a truncation of an antimatroid. \( \square \)

3 Non-simple greedoids

To describe the definitions of non-simple greedoids and non-simple antimatroids, we prepare some terminology. For a vector \( v \in \mathbb{R}^E \), \( |v| \) denotes its 1-norm \( \sum(|v_i| : i \in E) \). \( u \vee v \in \mathbb{R}^E \) denotes a join of vectors, i.e. \((u \vee v)_e = \max\{u_e, v_e\}\). For a word \( \alpha = a_1a_2 \cdots a_k \), a subsequence \( \alpha' = a_i, a_{i_2} \cdots a_{i_m} \) such that \( 1 \leq i_1 < i_2 < \cdots < i_m \leq k \) is a subword of \( \alpha \), and we write it as \( \alpha' \subseteq \alpha \). Let us denote by \( \alpha_x \) the number of repetitions of an element \( x \) in \( \alpha \), and \( [\alpha]_e = \alpha_e (e \in E) \), which is the score vector of \( \alpha \). An ascending chain of integral points \( u^{(0)} \leq u^{(1)} \leq \cdots \leq u^{(m)} \) \( (u^{(i)} \in \mathbb{Z}^E) \) is elementary if for \( i = 1, \ldots, m \), there exists \( j \in E \) such that \( u^{(i)}_j = u^{(i-1)}_j \) for \( j' \in E \) with \( j' \neq j \) and \( u^{(i)}_j = u^{(j-1)}_j + 1 \).

Björner and Ziegler [3] investigated three possible extensions of greedoids to non-simple languages such as

\[
\begin{align*}
(NG) & \quad \alpha, \beta \in L, |\alpha| > |\beta| \implies \exists x \in E : x \in \alpha \text{ and } \beta x \in L. \\
(SG) & \quad \alpha, \beta \in L, |\alpha| > |\beta| \implies \exists \text{subword } \alpha' \text{ of } \alpha : \beta \alpha' \in L \text{ and } |\alpha'| = |\alpha| - |\beta|. \\
(PG) & \quad \alpha, \beta \in L, |\alpha| > |\beta| \implies \exists x \in E : [\alpha]_x > |\beta|_x \text{ and } \beta x \in L.
\end{align*}
\]

where \( L \) is a left-hereditary non-simple language. They called \( L \) a 'non-simple greedoid' if it satisfies (NG), a 'strong greedoid' if it satisfies (SG), and a 'polygreedoid' if it satisfies (PG), respectively. They have mentioned that the conditions (SG) and (PG) are independent and that the reduced expressions of a finite Coxeter group gives rise to a strong greedoid (SG).

Instead of strong greedoids, in this paper, we shall consider polygreedoids as the definition of non-simple version of greedoids. That is, we shall call a left-hereditary non-simple language a non-simple greedoid if it satisfies (PG).

Let \( f \) be a real-valued function on \( E \times \mathbb{Z}_+ \times \mathbb{Z}_+ \), and suppose it satisfies

\[
\begin{align*}
&j = j', i \leq i' \implies f(e, j, i) \leq f(e, j', i'). 
\end{align*}
\]

And for \( \alpha = a_1 \cdots a_k \), we set \( F(\alpha) = f(a_k, [\alpha]_{a_k}, k) \). Then a generalized bottleneck function of (2.4) is generalized to

\[
\begin{align*}
w(\alpha) = \min_{i=1,\ldots,k} \{F(a_1 \cdots a_i)\} = \min_{i=1,\ldots,k} \{f(a_i, [\alpha]_{a_i}, i)\},
\end{align*}
\]

which we call a doubly generalized bottleneck function. Then Theorem 3.1 below holds, which seems to justify our choice of (PG) as the definition of non-simple greedoids.

Theorem 3.1 Let \( L \) be a finite non-simple language. If \( L \) is a non-simple greedoid, then for a doubly generalized bottleneck function, the algorithm GREEDY gives an optimal solution. Conversely, suppose \( L \) is left-hereditary and pure, and if the solution given by GREEDY is necessarily optimal for any doubly generalized bottleneck function, then \( L \) is a non-simple greedoid.
(Proof) We shall prove the first half. Let \( \alpha = a_1 \cdots a_k \) be the \( k \)-th intermediate solution of GREEDY. We use induction on \( k \). Suppose \( \alpha \) attains the maximum of \( w \) among the words of length \( k \) in \( L \), and \( a \alpha a_{k+1} \) to be the \((k+1)\)-th solution of GREEDY. And suppose, contrarily, \( a \alpha a_{k+1} \) is not optimal, and \( \beta = b_1 \cdots b_k b_{k+1} \) be an optimal solution of length \( k + 1 \). Then we have \( w(\beta) > w(a \alpha a_{k+1}) \). Hence,

\[
\min_{i=1, \ldots, k+1} \{ f(b_i, [\beta]_{b_i}, i) \} > \min_{i=1, \ldots, k+1} \{ f(a_i, [\alpha]_{a_i}, i) \}
\]

If the minimum of the right-hand side is attained for some \( i \) with \( 1 \leq i \leq k \), then we have

\[
\min_{i=1, \ldots, k} \{ f(b_i, [\beta]_{b_i}, i) \} \geq \min_{i=1, \ldots, k+1} \{ f(b_i, [\beta]_{b_i}, i) \} > \min_{i=1, \ldots, k+1} \{ f(a_i, [\alpha]_{a_i}, i) \} = \min_{i=1, \ldots, k} \{ f(a_i, [\alpha]_{a_i}, i) \},
\]

which contradicts the optimality of \( \alpha = a_1 \cdots a_k \). Hence the minimum is attained at \( i = k+1 \) and we have

\[
\min_{i=1, \ldots, k+1} \{ f(b_i, [\beta]_{b_i}, i) \} > f(a_{k+1}, [\alpha]_{a_{k+1}}, k+1).
\]

Since \( |\beta| > |\alpha| \), (PG) implies that there exists \( x \in \tilde{\beta} \) such that \( \beta_x > \alpha_x \) and \( \alpha x \in L \). Hence \( x \) appears in \( \beta \) more than \( \alpha_x \) times. So let \( b_j \) be the \((\alpha_x + 1)\)-th \( x \) in the sequence \( \beta \). Then, from the monotonicity of \( f \), we have

\[
F(ab_j) = F(ax) = f(x, \alpha_x + 1, k + 1)
\]

\[
\geq f(x, \alpha_x + 1, j) = F(b_1 \cdots b_j)
\]

\[
\geq \min_{i=1, \ldots, k+1} \{ F(b_1 \cdots b_i) \} > F(\alpha a_{k+1})
\]

which contradicts the choice of \( a_{k+1} \). Hence the proof of the first half is completed.

We shall describe the proof of the second half. Suppose, contrarily, that (PG) does not hold for \( \alpha = a_1 \cdots a_n, \beta = b_1 \cdots b_m \in L \) with \( |\alpha| > |\beta| \). Take \( k \) to be the minimal index such that \( [a_1 \cdots a_k] > [\beta_{ak}] \), and let \( x = a_k \). By assumption such \( k \) necessarily exists. Let us define a function on \( E \times \mathbb{Z}_+ \times \mathbb{Z}_+ \) such that

\[
f(e, j, i) = \begin{cases} 
1 & \text{if either } j \leq \beta_z \text{ or } i > \beta_x \ 
0 & \text{otherwise}
\end{cases}
\]

It is easy to see that \( f \) satisfies (3.1). Using this \( f \), let us define

\[
F(y_1 \cdots y_p) = f(y_p, [y_1 \cdots y_{p-1}], [y_1 \cdots y_p]_{z}) \quad \text{for } y_1 \cdots y_p \in L, \quad (3.3)
\]

and \( w \) be the associated bottleneck function.

Then \( w(\alpha) = 1 \) holds. Actually, for \( i = 1, \ldots, k - 1 \), we have \( [a_1 \cdots a_i] \leq \beta_{a_i} \) from the choice of \( k \), and hence \( F(a_1 \cdots a_i) = 1 \). For \( i = k, \ldots, n \), \( [a_1 \cdots a_i]_{x} \geq [a_1 \cdots a_k]_{x} > \beta_x \) and so \( F(a_1 \cdots a_i) = 1 \). Hence \( w(\alpha) = \min \{ F(a_1 \cdots a_i) : i = 1, \ldots, n \} = 1 \). In particular, this implies that \( \alpha \) is an intermediate solution and will be extended to a final solution \( \alpha^1 \) of GREEDY. A similar argument shows \( w(\alpha^1) = 1 \).

For \( j = 1, \ldots, m \), we have \( f(b_j, [b_1 \cdots b_j]_{b_j}, [b_1 \cdots b_j]_{x}) = 1 \) since \( [b_1 \cdots b_j]_{b_j} \leq \beta_{b_j} \) trivially holds. Hence \( w(\beta) = 1 \), and \( \beta \) is also an intermediate solution of GREEDY, which will be extended to a final solution \( \beta^1 \). Suppose \( z \in E, \beta z \in L \). Then by assumption, \( z \neq x \). Since \( [\beta]_z = \beta z + 1 > \beta z \) and \( [\beta]_x = \beta x, f(z, [\beta]_z, [\beta]_x) = 0 \). From the definition of \( w \), \( w(\beta^1) = 0 \) follows. Hence \( w(\alpha^1) \neq w(\beta^1) \), which contradicts the optimality of the solutions by GREEDY. And the proof is completed.
4 Non-simple antimatroids

Björner, Lovász and Shor [2] proposed that a non-simple antimatroid is defined as a language such that it
is left-hereditary, locally free and permutable. That is, let \( L \) be a left-hereditary non-simple language on
\( E \). Then \( L \) is said to be a non-simple antimatroid if it holds that

\[
\text{LF} \quad \alpha x, \alpha y \in L, \ x \neq y \ (x, y \in E) \implies \alpha yx \in L. \\
\text{PM} \quad \alpha, \beta \in L, \ [\alpha] = [\beta], \ \alpha x \in L \ (x \in E) \implies \beta x \in L.
\]

The pair of conditions (LF) and (PM) is easily seen to be equivalent to the exchange property (EX) below.

\[
\text{EX} \quad \alpha x, \beta \in L, \ [\alpha] \leq [\beta], \ [\alpha x] \not\equiv [\beta], \ (x \in E), \implies \beta x \in L.
\]

Furthermore, (EX) is equivalent to the stronger exchange property (StEX). (See [2].)

\[
\text{StEX} \quad \alpha, \beta \in L \implies \exists \text{ subword } \alpha' \text{ of } \alpha : \beta \alpha' \in L, \ [\beta \alpha'] = [\alpha] \lor [\beta].
\]

Let us denote by \( \mathcal{N} = \{ [\alpha] : \alpha \in L \} \) the set of score vectors of \( L \). It follows immediately from the
above properties of \( L \) that \( \mathcal{N} \) satisfies the following:

1. \( 0 \in \mathcal{N} \),
2. For any \( v \in \mathcal{N} \), there exists an elementary chain in \( \mathcal{N} \) from the origin \( 0 \) to \( v \),
3. if \( u, v \in \mathcal{N} \), then \( u \lor v \in \mathcal{N} \).

In particular, with respect to the ordinary partial order among vectors, \( \mathcal{N} \) constitutes a locally free lattice.
Hence, it is isomorphic to a lattice of feasible sets of a certain simple antimatroid.

Conversely, we shall call a finite collection of non-negative integral points a score space if it satisfies
the above three conditions. Every score space is derived from a non-simple antimatroid. In fact,

\[
L_{\mathcal{N}} = \{ a_1 \cdots a_k : a_i \in E, \ 0 \leq [a_1] \leq [a_1a_2] \leq \cdots \leq [a_1 \cdots a_k] \text{ is a maximal chain of } \mathcal{N} \} \quad (4.1)
\]

is a non-simple antimatroid whose set of score vectors equals to \( \mathcal{N} \). Hence the notions of non-simple
antimatroid and score space are seen to be equivalent.

A score space has the strong accessibility mentioned below.

**Proposition 4.1** Let \( u, v \in \mathcal{N} \) and \( u \leq v \). Then there exists an elementary chain in \( \mathcal{N} \) connecting \( u \) and \( v \).

(Proof) From \( v \in \mathcal{N} \), there exists an elementary chain \( \{ v_i \} \) from the origin to \( v \). Then \( \{ u \lor v_i \} \) is a chain
in \( \mathcal{N} \) possibly including repeated elements, which contains obviously an elementary chain between \( u \) and
\( v \) as a subsequence. \( \square \)

This can be restated in terms of language as

**Corollary 4.1** Let \( L \) be a non-simple antimatroid, \( \alpha, \beta \in L \), and suppose \( [\alpha] \leq [\beta] \). Then there exists a
subword \( \gamma \) of \( \beta \) such that \( \alpha \gamma \in L \) and \( [\alpha \gamma] = [\beta] \).
Let $L$ be a non-simple antimatroid, and $N$ be the score space of $L$. Suppose $f$ to be a real-valued function on $E \times N$ such that

$$u_e = v_e, \ u \leq v \implies f(e, u) \leq f(e, v), \ (4.2)$$

Then a nested bottleneck function of (2.6) is generalized to

$$w(\alpha) = \min \{ f(a_i, [a_1 \cdots a_i]) : i = 1, \ldots, r \} \quad (\alpha = a_1 a_2 \cdots a_r \in L), \ (4.3)$$

which we shall call a score-nested bottleneck function. Then Theorem 2.2 is extended to non-simple antimatroids as follows.

**Theorem 4.1** For a non-simple antimatroid and a score-nested bottleneck function, the solution by GREEDY is necessarily optimal. And for a left-hereditary non-simple pure language $L$, if the greedy solution is optimal for any score-nested bottleneck function, then $L$ is a truncation of a non-simple antimatroid.

**(Proof)** Let $\alpha = x_1 \cdots x_k$ be the greedy solution after the $k$-th iteration of GREEDY. We use induction on $k$. Suppose that the assertion holds until $k$, and let $\alpha x_{k+1}$ be the greedy solution of length $k+1$. Suppose there exists an optimal solution $\beta y_{k+1} = y_1 \cdots y_k y_{k+1} \in L$ of length $k+1$ and $w(\beta y_{k+1}) > w(\alpha x_{k+1})$. By the assumption and the induction hypothesis, we have

$$\min_{i=1, \ldots, k+1} \{ f(y_i, [y_1 \cdots y_i]) \} > f(x_{k+1}, [x_1 \cdots x_{k+1}]) \quad (4.4)$$

If $\alpha = \beta$ holds, then from the definition of GREEDY, we have

$$w(\beta y_{k+1}) > w(\alpha x_{k+1}) \geq w(\alpha y_{k+1}) = w(\beta y_{k+1}), \quad (4.5)$$

which is a contradiction.

Hence we have $\alpha \neq \beta$. Then there exists $p$ with $1 \leq p \leq k$ such that

$$[y_1 \cdots y_{p-1}] \leq [\alpha], \quad [y_1 \cdots y_p] \notin [\alpha]. \quad (4.6)$$

By (EX), we have $\alpha y_p \in L$.

Clearly, $[\alpha y_p] \geq [y_1 \cdots y_p]$ and $[\alpha y_p]_{y_p} = [y_1 \cdots y_p]_{y_p}$. Hence from the 2-monotonicity of $f$, we have

$$f(y_p, [\alpha y_p]) \geq f(y_p, [y_1 \cdots y_p]) \geq \min_{i=1, \ldots, k+1} \{ f(y_i, [y_1 \cdots y_i]) \} > f(x_{k+1}, [\alpha x_{k+1}]),$$

which contradicts the choice of $x_{k+1}$. The proof of the first half is completed.

Conversely, let $L$ be a left-hereditary non-simple language and $\rho(L) < +\infty$. Suppose GREEDY always gives an optimal solution.

We shall first prove $L$ is locally free. Let $\alpha \in L$ and $|\alpha| < \rho(L) - 1$. Suppose $\alpha x, \alpha y \in L$, $x \neq y$ and $\alpha y x \notin L$. Let us define

$$f(e, v) = \begin{cases} 1 & \text{if } v_e \leq [\alpha y]_e \text{ or } v_x > \alpha_x \\ 0 & \text{otherwise} \end{cases} \quad (e \in E, \ v \in \mathbb{Z}^E)$$

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We shall show that $f$ satisfies (4.2). Suppose, contrarily, $v_e = v'_e$, $v \leq v'$, and $1 = f(e, v) > f(e, v') = 0$. Then either $v_e \leq [ay]_e$ or $v_x \geq \alpha_x + 1$ holds, and both $v'_e > [ay]_e$ and $v'_x < \alpha_x + 1$ hold. These give a contradiction. Hence $f$ satisfies (4.2). For this $f$, we have $w(ax) = w(ay) = 1$. So $ax$, $ay$ arise as an intermediate solution of GREEDY, and they are extended to the final solutions denoted $\alpha^1$ and $\alpha^2$, respectively.

Let $\alpha^1 = axu_1 \cdots u_q$. For $i = 1, \ldots, q$, $[axu_1 \cdots u_i]_x > \alpha_x$ and $f(u_i, axu_1 \cdots u_i) = 1$ hold. Hence $w(\alpha) = 1$. Next we shall show $w(\alpha^2) = 0$. Suppose $\alpha y z \in L$ for $z \in E$. By assumption, $z \neq x$. Trivially, $[\alpha y z]_z \geq [\alpha y]_z + 1 > [\alpha y]_z$ and since $y \neq x$ and $z \neq x$, we have $[\alpha y z]_x = \alpha_x$. Hence $f(z, \alpha y z) = 0$, and $w(\alpha^2) = 0$ follows. Since $w(\alpha^1) \neq w(\alpha^2)$, this contradicts the optimality of the algorithm GREEDY.

Next we shall show the permutability of $L$. Let $\alpha, \beta \in L$ and $|\alpha| = |\beta| < \rho(L)$. Suppose $[\alpha] = [\beta]$, $\alpha x \in L$ and $\beta x \notin L$. Then

$$f'(e, v) = \begin{cases} 1 & \text{if } v_e \leq \alpha_e \text{ or } v_x > \alpha_x \\ 0 & \text{otherwise.} \end{cases}$$

is a function satisfying (4.2). Let us first show $w(ax) = 1$ and $w(\beta) = 1$. Suppose $\alpha = a_1 \cdots a_n$. For $i = 1, \ldots, n$, since $[a_1 \cdots a_i]_{a_i} \leq [\alpha]_{a_i}$ is trivially satisfied, we have $f'(a_i, a_1 \cdots a_i) = 1$. Since $[ax]_x > \alpha_x$, we have $f(x, ax) = 1$. Hence $w(ax) = 1$ follows. Suppose $\beta = b_1 \cdots b_n$. Since $[b_1 \cdots b_i]_{b_i} \geq [\beta]_{b_i} = [\alpha]_{b_i}$ is obvious, $f'(b_i, b_1 \cdots b_i) = 1$ holds for $i = 1, \ldots, n$. Hence $w(\beta) = w(b_1 \cdots b_n) = 1$. Both $ax$ and $\beta$ are intermediate solutions of GREEDY, and they are extended to the final solutions denoted $\alpha^1$ and $\beta^1$, respectively.

Suppose $\alpha^1 = axd_1 \cdots d_r$. Then for any $j = 1, \ldots, r$, we have $f'(d_j, axd_1 \cdots d_j) = 1$ since $[axd_1 \cdots d_j]_x > [\alpha]_x$ obviously holds. Hence we have $w(\alpha^1) = 1$. Next we shall show $w(\beta^1) = 0$. Suppose $\beta z \in L$ for $z \in E$. From assumption, $z \neq x$. Then $[\beta z]_z = \beta_z + 1 = \alpha_z + 1 \not\leq \alpha_x$ and $[\beta z]_x = \beta_x = \alpha_x \not> \alpha_x$, which gives $f(z, \beta z) = 0$. Hence by definition, $w(\beta^1) = 0$. This contradicts the optimality of GREEDY, and the proof is completed.

As a concluding remark, we shall present a variant of Theorem 4.1. Replacing the partial order among score vectors in the exchange property (EX) with that of subword-inclusion provides a condition

(UI) \quad $\alpha, \beta \in L$, $\alpha \subseteq \beta$, $\alpha x \not\subseteq \beta$ \implies $\beta x \in L$.

We can consider it as an upper interval property over non-simple words. If a non-simple language $L$ is left-hereditary and satisfies (UI), let us call it an upper interval language. As is easy to observe, an upper interval language is locally free, but not permutable in general.

A complete analogue of Theorem 4.1 holds for upper interval languages. Let $f$ be a real-valued function on $E \times L$, and suppose $f$ satisfies

$$\alpha e = \alpha'_e, \quad \alpha \subseteq \alpha' \implies f(e, \alpha) \leq f(e, \alpha').$$

This gives rise to a bottleneck objective function

$$w(\alpha) = \min_{i=1,\ldots,r} \{ f(a_i, a_1 \cdots a_i) : i = 1, \ldots, r \},$$

and we have
Theorem 4.2 Let \( L \) be a non-simple language which is left-hereditary and pure. If \( L \) is a truncation of an upper interval language, then GREEDY gives an optimal solution for any function of (4.8). Conversely, if the solution given by GREEDY is always optimal, then \( L \) is a truncation of an upper interval language.

(Proof) The proof is completely analogous to that of Theorem 4.1. \( \square \)

References


