Antimatroids and Convex Geometries
– The fundamentals

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An antimatroid and a convex geometry are completely equivalent concepts since they are just the complement of each other. A convex geometry is a closure system whose closure function meets the anti-exchange property whereas the collection of closed sets of a matroid is another closure system whose closure function satisfies the exchange property. Hence both could be seen as subclasses of closure systems.

This monograph is intended to gather and survey the studies of antimatroids/convex geometries, and closure systems, especially in the applications in computer science. A closure system gives a perspective viewpoint over all of these.

The choice function theory in mathematical social science is also treated here as a path-independent choice function is known to be cryptomorphic to an extreme function of a convex geometry. There is found a great wide framework which completely clarifies the relation between choice functions and extensive functions based on the idea of neighbourhoods.

The notion of closure systems appears separately in many areas such as implicational system (functional dependencies in relational database theory), formal concept analysis, knowledge space, logic, and so on. But they are not yet merged into one. There is however a mighty key word ‘closure operator’ common to all. We have a hope therefore, in view of tomorrow, for tomorrow will have its own developments.

K. K. and M. N.
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### 19.1 Proofs of Exercises

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Chapter 1

Introduction

The Master said, ‘Is it not a pleasure to learn and practice what is learned?
Is it not a joy to have friends come together from far?
Is it not gentlemanly not to take offence though men may take no note of him? ❧
– The Analects of Confucius

1.1 The review of this monograph

An antimatroid was first studied by Korte and Lovász (1981 [108], 1984 [109]) under various names such as a shelling structure, an alternative precedence structure or an APS-greedoid [21, 109]. Antimatroids typically arise from shelling on various combinatorial objects such as finite sets of points in an affine space, posets, trees, chordal graphs, digraphs, and so on.

Edelman [61] established the correspondence between the meet-distributive lattices and the antirexchange property of closure operators. The systematic study of convex geometry seems to be started by Jamison [96], and Edelman and Jamison [63]. An antimatroid and a convex geometry are completely equivalent notions since they are just the complement of each other.

A closure systems is the collection of subsets of a set that contains the entire set and is closed under intersection.

There are two possible common generalizations of matroid and convex geometry/antimatroid. A convex geometry is a special case of closure systems such that the associated closure function satisfies the antirexchange property. An antimatroid is the collection of complements of elements of a convex geometry. The closed sets (flats) of a matroid form a closure system as well. In terms of closure system, a matroid is a closure system whose closure function meets the Steinitz-McLane exchange property. Both of them can be seen as subclasses of closure systems. An antimatroid and the set of independent sets of a matroid are special cases of greedoids. Hence they are special cases of greedoids as well.

We emphasize in this monograph that the viewpoint of closure system is crucial to the theory of convex geometry/antimatroid. Even in closure systems, we can naturally define the familiar concepts such as independent sets, dependent sets, circuits, and so on. Also in a matroid, an independent set and a circuit defined from a closure function are the same with the ordinary definitions.

Various subjects are modeled by closure systems such as implicational system, formal concept analysis, knowledge space, logic and so on.
In a lattice-theoretic view, the notion of a convex geometry is cryptomorphic to a lower locally distributive lattice. The study of the class of lower locally distributive lattices can be traced back to Dilworth’s work \[52\] in 1940. This class has ever been rediscovered independently several times, and has different names: Meet-distributive lattice (Edelman 1980), lower semidistributive lattice (Avann 1961), α–affaibi (Boulaye 1968), G–lattice by (Pfaltz 1969), lower locally distributive lattice (Greene and Markowsky 1974), and so on. The historical story is found in the short note of B. Monjardet \[123\].

In the theory of choice functions of mathematical social science, Plott (1973) introduced the notion of path-independent choice functions as a weakening of rationalizable choice functions, and Koshevoy (1999) discovered that a path-independent choice function is cryptomorphic to a convex geometry. More precisely, a choice function is path-independent if and only if it is an extreme function of closure function of a convex geometry. This connection between choice functions and closure functions is thoroughly generalized by Danilov and Koshevoy (2009) in a wide framework introducing the concept of neighborhoods.

For the basic notions and the examples of convex geometries and antimatroids, the readers can refer to Korte, Lovász and Schrader \[112\], Edelman and Jamison \[63\], and Björner and Ziegler \[28\]. Greedoids and antimatroids are thoroughly investigated as languages in Björner \[21\].

The largest part of this monograph is given to the collection of axiom sets, the collection of the examples, and the summaries of so far known results concerning closes convex geometries and closure systems so that it would help the readers study further in this area.
Chapter 2

Mathematical Preliminaries

2.1 Set systems

We begin with describing some definitions about set families. Let $E$ be a set. $2^E$ is the collection of all the subsets of $E$, i.e. the power set of $E$. For $S \subseteq 2^E$, we call $(S, E)$ a set system. $S^\ast = \{E \setminus A : A \in S\}$ is the collection of the complements of the elements of $S$. Let $(S_1, E_1)$ and $(S_2, E_2)$ be two set systems with $E_1 \cap E_2 = \emptyset$. Then their direct-sum $(S_1 \oplus S_2, E_1 \cup E_2)$ is defined by $S_1 \oplus S_2 = \{A \cup B : A \in S_1, B \in S_2\}$.

A set system $(S, E)$ is called a laminar if for every pair of elements of $S$, either one contains another or they are disjoint.

For a subset $A \subseteq E$, we shall define collections of subsets as follows.

$$
S/A = \{X \setminus A : X \in S, A \subseteq X\},
S \setminus A = \{X : X \in S, X \cap A = \emptyset\},
S - A = \{X \setminus A : X \in S\}.
$$

$S/A$ is called the contraction by $A$, and $S \setminus A$ is the deletion by $A$. $S - A$ is the trace of $S$ on $E - A$. Then $S^\ast \setminus A = \{A \cup X : X \in S/A\}^\ast$ holds for any $A \subseteq E$. Furthermore, when $A \cap B = \emptyset$, $(S/A) \setminus B = (S \setminus B)/A$ holds. After repeating the deletion and contraction, the resultant family of sets is called a minor. The trace $S - A$ for $A \subseteq E$ is also called a trace-minor.

In this note a rooted set on $E$ means a pair $(X, e)$ of a subset $X \subseteq E - e$ and an element $e \in E - X$. It was first introduced in [101]. Rooted sets appear in several places in this literature as a rooted circuit, a rooted cocircuit, and so on.

We remark that our notation of a rooted set is slightly distinct from the definition of other researchers. For instance, a rooted circuit was so far defined as a pair $(C, e)$ with $e \in C$ in [51, 112] etc. In our term, $(C \setminus e, e)$ is called a rooted circuit. You will see the merit of our notation, for instance, in implicational theory of Section 17.3. There a rooted circuit $(X, e)$ of our style directly corresponds to an implication $X \rightarrow e$.

2.2 Independence systems and simplicial complexes

A collection $\mathcal{I}$ of subsets of $E$ is an independence system if any subset of an element of $\mathcal{I}$ also belongs to $\mathcal{I}$. Given an independence system $\mathcal{I}$, a set in $\mathcal{I}$ is said to be an independent set. Otherwise it is called a
dependent set. A minimal dependent set is called a circuit. By definition, a set is independent if and only if it contains no circuit. Any deletion and any contraction of an independence system are independence systems.

An independence system is called a simplicial complex in topology. An element of a simplicial complex is called a face. A maximal face is called a facet. If all the facets are of the same size, a simplicial complex is called pure.

If a subfamily of a simplicial complex $\Delta$ is a simplicial complex in its own right, it is called a subcomplex of $\Delta$. For a simplicial complex $\Delta$ and a face $F \in \Delta$, $|F| - 1$ is called the dimension. The collection of faces of $\Delta$ such that their dimensions are at most $k$ forms a subcomplex, called the $k$-skeleton of $\Delta$ and denoted by $\Delta^k$. A 1-skeleton is equivalent to a simple undirected graph.

We shall explain some notations in the theory of topology. Let $\Delta$ be a simplicial complex on $E$. For $A \in \Delta$ and $B \subseteq A$, $\text{Del}_\Delta(A) = \{ B \in \Delta : A \cap B = \emptyset \}$ is the deletion by $A$. In case of simplicial complexes, the deletion and the trace coincide with each other, namely, $\Delta \setminus A = \Delta - A$. $\text{Link}_\Delta(A) = \{ B \in \Delta : A \cap B = \emptyset, A \cup B \in \Delta \}$ is called the link of $A$. $\text{Link}_\Delta(A)$ is equal to the contraction $\Delta / A$ by $A$ in our term.

### 2.3 Clutters and blockers

A clutter $C$ is a collection of sets such that any member does not properly contain another member, i.e. there exist no $A, B \in C$ such that $A \subsetneq B$. Note that the collection of minimal sets of a family of sets necessarily constitutes a clutter. Hence the collection of circuits of an independence system is a clutter.

For a given clutter $C$, a set is named a transversal if it intersects with every element of $C$. The collection of minimal transversals forms a clutter which is called the blocker of $C$ and denoted by $b(C)$. By definition, the blocker is necessarily a clutter. In particular, if $C = \{ \emptyset \}$, then $b(C) = \emptyset$.

**Proposition 2.1** For an arbitrary clutter $C$, $b(b(C)) = C$ holds.

(Proof) For a family $H \subseteq 2^E$, $H \uparrow$ denotes $\{ Y \in 2^E : X \subseteq Y \text{ for some } X \in H \}$. In order to prove $b(b(L)) = L$, it is sufficient to show $b(b(L)) \uparrow = L \uparrow$.

Suppose $X \in L \uparrow$. By definition, $X$ intersects every member of $b(H)$. Hence $X \in b(b(L)) \uparrow$.

Conversely suppose $X \notin L \uparrow$. If $E \setminus X$ does not intersect a certain element $Z \in L$, $X$ includes $Z$, which is a contradiction. Hence $E \setminus X$ intersects every element of $L$, and so $E \setminus X \in b(L) \uparrow$. Hence there is an element $W \in b(L)$ such that $W \subseteq E \setminus X$, which readily implies $X \cap W = \emptyset$ and $X \notin b(b(L)) \uparrow$. This completes the proof. $\square$

The packing number of a clutter is the maximum size of a family of members of a clutter such that any pair of them does not intersect. Clearly, the packing number of a clutter $C$ is equal to or less than the minimum size of the elements of the blocker $b(C)$. If these two numbers are equal, we say that $C$ packs.

**Example 2.2** Examples of clutters that do not pack.

1. $Q_6 = \{\{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{2, 4, 5\}\}$: Actually the packing number of $Q_6$ is one, and the minimum size of the transversals of $Q_6$ is two. (In contrast, $b(Q_6) = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{1, 3, 5\}, \{2, 3, 6\}, \{1, 4, 6\}, \{2, 4, 5\}\}$ packs).
(2) \( C_s = \{\{1, 2\}, \{2, 3\}, \ldots, \{s-1, s\}, \{s, 1\}\} \) \( s \geq 3, \) odd: The packing number of \( C_s \) is \((s - 1)/2\), and the minimum size of the transversals of \( C_s \) is \((s + 1)/2\).

In case of clutters, their deletions and contractions are defined in a little bit different way. For a clutter \( C \) and \( A \subseteq E \), the contraction and the deletion of a clutter \( C \) by \( A \) are defined as

\[
C/A = \text{Minimal}(\{X \setminus A : X \in C\}), \quad C \setminus A = \{X \in C : X \cap A = \emptyset\},
\]

respectively, where \text{Minimal}( ) denotes the collection of minimal sets of a set family ( minimal with respect to inclusion relation). After repeating deletions and contractions, the resultant clutter is called a \textit{minor} of a clutter \( C \).

The operations of contraction and deletion are inversed for the blocker of a clutter.

\[
b(C/A) = C \setminus A, \quad b(C \setminus A) = C/A.
\]

Let \( C \subseteq 2^E \) be a clutter, and \( M \) to be the \( 0, 1 \)-matrix associated with \( C \) with the size \(|C| \times |E|\), i.e. each row vector of \( M \) is the characteristic vector of an element in \( C \). Suppose \( w \) to be a nonnegative integral row vector, and we consider the following linear optimization problem.

\[
(P) \begin{cases} 
\min & w \cdot x \\
x \geq 0 \\
Mx \geq 1
\end{cases} \quad (D) \begin{cases} 
\max & y \cdot 1 \\
y \geq 0 \\
yM \leq w
\end{cases}
\]

The property that a clutter packs can be defined in terms of the linear programming.

- When \( w = 1 \), \( C \) is said to pack if (P) and (D) both have integral optimal solutions.
- \( C \) is ideal if (P) has an integral optimal solution for every \( w \in \{0, 1, +\infty\}^n \). (It is equivalent to say that (P) has an integral optimal solution for an arbitrary non-negative integer valued \( w = Z^n_+ \).)
- If (P) and (D) both have integral optimal solution for every \( w \in \{0, 1, +\infty\}^n \), \( C \) is said to have the \textit{packing property}. (This is equivalent to the condition that every minor of \( C \) packs.)
- If (P) and (D) have integral optimal solutions for every \( w \in Z^n_+ \), \( C \) is said to have the \textit{Max-Flow Min-Cut property}. In that case, we also simply say that \( C \) is MFMC.

These properties follow the relation below.

\[
\text{MFMC} \implies \text{packing property} \implies \text{C packs}
\]

The idealness, the MFMC property and the packing property are closed under taking the operation of minors.

**Proposition 2.3** For a clutter \( C \), the following hold.

(1) If \( C \) is ideal, then every minor is ideal.

(2) If \( C \) is MFMC, then every minor is MFMC.

(3) If \( C \) is ideal, then \( b(C) \) is also ideal. ( Lehman’s Theorem )

**Example 2.4** \( Q_6 \) is ideal, but does not pack, while \( b(Q_6) \) is MFMC.

2.4 Affine spaces and affine point configurations
Chapter 3

Graphs

3.1 Graphs and hypergraphs

An undirected graph $G$ is a pair of a vertex set $V(G)$ and an edge set $E(G)$. Each edge $e \in E(G)$ has the set of endpoints $\partial(e) = \{u, v\}$ ($u, v \in V$). For an edge $e \in E(G)$, if $|\partial(e)| = 1$, $e$ is called a loop. For two edges $e, f \in E(G)$, if $\partial(e) = \partial(f)$, they are called parallel. A graph is trivial if $|V(G)| = 1$ and $|E(G)| = \emptyset$. A graph is simple if it has neither loops nor parallel edges. In a simple undirected graph, an edge $e$ is identified with a set $\{u, v\}$. Then $E(G)$ is equal to a collection of two-element subsets of $V(G)$. When notifying nothing, a graph implies an undirected simple graph.

We can extend the notion of graphs to that of hypergraphs. A hypergraph is nothing but a set system. More exactly, a hypergraph $\mathcal{H}$ is a pair $(V, \mathcal{E})$ where $V$ is a nonempty finite set, and $\mathcal{E}$ is a collection of nonempty subsets of $V$. Suppose that $\mathcal{E} = \{E_i : i \in I = \{1, 2, \ldots, n\}\}$. For $J \subseteq I$, $(V, \{E_j ; j \in J\})$ is said to be a partial hypergraphs. A subhypergraph induced by $U \subseteq V$ is $(U, \{E_k \cap U : E_k \cap U \neq \emptyset, k \in I\})$. Namely, a subhypergraph is a hypergraph with some vertices removed.

Two vertices $u, v$ of $G$ are adjacent if there exists an edge $e$ with $\partial(e) = \{u, v\}$. A vertex is called an isolated vertex if there is no vertex adjacent to it. A vertex $v$ and an edge $e$ is incident if $v \in \partial(e)$. The neighbourhood of a vertex $u$ is a set $N(v) = \{u \cup \{w : \{u, w\} \in E(G)\}$. A path in a graph is a sequence $v_0v_1 \cdots v_k$ of distinct vertices such that $v_{i-1}$ and $v_i$ are adjacent for each $1 \leq i \leq k$. A cycle is a path $v_0v_1 \cdots v_m$ ($m \geq 3$) such that $v_m$ and $v_0$ are adjacent. A chord of a cycle $v_0v_1 \cdots v_m$ is an edge $v_iv_j$ such that $i + 1 < j$ except when $i = 0, j = m$. If a path has no chord, it is called a chordless path. A chordless cycle is similarly defined.

For a subset $W$ of the vertex set $V(G)$, we define $\delta(W) = \{\{u, v\} \in E(G) : u \in W, v \in V(G) - W\}$. If $\delta(W)$ is nonempty, we shall call it a cut-set. A minimal cut-set is called a cocycle, where ‘minimal’ refers to inclusion relation.

For a subset $A$ of $V(G)$, the induced subgraph $G_A$ is a graph with the vertex set $A$ and the edge set $E(G_A) = \{\{u, v\} \in E(G) : u, v \in A\}$. $G_A$ is sometimes denoted by $G[A]$. For an edge subset $B$ of $G$, $G|B$ is a graph such that the edge set is $B$ and the vertex set is the set of vertices incident to $B$ in $G$. $G|B$ is called a subgraph.

A graph is connected if there exists a path between every pair of the vertices. A maximal connected subgraph is called a connected component of a graph. In general, a graph consists of several connected components. A subset $S$ of vertices of a graph $G$ is a vertex separator if deleting it from the graph increases
the number of connected components. That is, there is a nontrivial partition \( A \cup B \cup S = V(G) \), and for every pair of vertices \( a \in A \) and \( b \in B \), every path connecting \( a \) and \( b \) passes through at least one vertex in \( S \). In such a case, \( S \) is called an \( a,b \)-vertex separator. A set of vertices of a graph is a clique if every pair of vertices in it are adjacent.

A simple graph with \( n \) vertices is a complete graph, denoted by \( K_n \), if all the vertices are adjacent to each other. A graph \( G \) is bipartite if there is a non-trivial partition of the vertex set \( V(G) \) into two nonempty classes \( V_1 \cup V_2 \) such that every edge has one endpoint in \( V_1 \) and another in \( V_2 \). It is easy to observe that a graph is a bipartite graph if and only if it contains a cycle of odd length. For a bipartite graph, if every pair of vertices in \( V_1 \) and \( V_2 \) are adjacent, it is called a complete bipartite graph, denoted by \( K_{m,n} \), where \( m = |V_1| \) and \( n = |V_2| \). In particular, \( K_{1,3} \) is called a claw. A graph \( G \) is called claw-free if it includes no induced subgraph isomorphic to \( K_{1,3} \).

For any graph \( H \), let \( L(H) \) be a graph such that the vertex set of \( L(H) \) is the edge set of \( H \) and two vertices in \( L(H) \) are joined by an edge if the corresponding edges in \( H \) are adjacent. A graph \( G \) is said to be a line graph if it is isomorphic to \( L(H) \) for some graph \( H \).

It is well known (Beineke [11]) that a graph is a line graph if and only if it contains no induced subgraph isomorphic to one of the certain nine graphs. \( K_{1,3} \) is one of the nine forbidden induced subgraph. That is, a line graph should be claw-free.

For instance, the complete graph \( K_5 \) and the complete bipartite graph \( K_{3,3} \) are shown in Fig. 3.1 and Fig. 3.2.

A series-extension of a graph is putting a new vertex on an edge, which divides the edge to two edges. A parallel-extension of a graph is attaching a new edge with the same endpoints of an existent edge, so that the number of edges is increased by one. A series-parallel graph is the resultant graph starting from one edge by repeating series-extensions and parallel-extensions. Naturally a series-parallel graph is not simple in general.

A directed graph (or shortly a digraph) \( G \) in this monograph is a pair \((V(G), E(G))\) such that \( E(G) \subseteq V(G) \times V(G) \). A digraph is acyclic if it has no directed circuit. A transitively closed graph is a digraph such that for each \((e, f) \in E(G)\) and \((f, g) \in E(G)\), \((e, g)\) must be in \( E(G) \). Clearly, for a partial order \( \mathcal{R} \subseteq V \times V \) on a non-empty finite set \( V \), \((V, \mathcal{R})\) is a transitively closed acyclic graph, and vice versa.

For a subset \( W \neq \emptyset \) of the vertex set \( V(G) \), \( \delta^+(W) \) denotes the set of edges from a vertex in \( W \) to a vertex in \( V(G) - W \), i.e. \( \delta^+(W) = \{(u, v) : u \in W, v \in V(G) - W\} \). If \( \delta(W) \neq \emptyset \) and \( \delta(V(G) - W) = \emptyset \), then \( \delta(W) \) is called a directed cut-set.

A graph in this monograph is an undirected finite simple graph unless otherwise stated.

![Figure 3.1: The complete graph \( K_5 \)](image)

![Figure 3.2: The complete bipartite graph \( K_{3,3} \)](image)
3.2 Forbidden-minors of graph classes

For an edge $e$ in a graph $G$, $G/e$ is the graph obtained by shrinking and deleting $e$ and identifying the two end vertices of $e$. This operation is called the contraction of $e$. $G - e$ denotes the subgraph $G|(E - e)$. This is called a deletion of an edge $e$. The graph obtained from $G$ by repeating the deletion and the contraction of edges is called a minor of $G$.

A property is called minor-hereditary provided that if a graph $G$ has the property, every minor of $G$ also satisfies it. For example, if a graph is planar, every minor is obviously planar. For a minor-hereditary property (P), a graph $G$ is called a forbidden graph if $G$ has not the property (P) but every proper minor of $G$ satisfies (P). A graph is planar if it can be drawn on a plane without crossing edges. Fig. 3.3 shows two examples of the drawings of the complete graph $K_4$. Obviously $K_4$ is a planar graph. Typical examples of non-planar graphs are $K_5$ and $K_{3,3}$. Actually they are the forbidden graphs of the class of planar graphs. The following is a variation of the well-known Kuratowski’s theorem.

**Theorem 3.1** A graph is planar if and only if it contains neither $K_5$ nor $K_{3,3}$ as a minor.

$K_4$ is the forbidden graph of series-parallel graphs.

**Theorem 3.2** A graph is a series-parallel graph if and only if it does not contain $K_4$ as a minor.

The theorems like Theorem 3.1 and 3.2 are called ‘forbidden-minor theorems.’

3.3 Chordal graphs

A graph is a chordal graph if any cycle of length at least four has necessarily a chord.

**Theorem 3.3 ([80])** Let $G$ be a graph. Then $G$ is a chordal graph if and only if every minimal vertex separator is a clique.

(Proof)

(⇒) Suppose $G$ is a chordal graph, and $S \subseteq V = V(G)$ be a minimal vertex separator. $G_{V - S}$ is divided into two induced subgraphs $G_A$ and $G_B$ where $A \cup B \cup S = V$ is a nontrivial partition of $V$. Take any two vertices $u, v$ in $S$, and we shall show that $u$ and $v$ are adjacent. Suppose contrarily that it doesn’t hold. Then there exists a path $P$ between $u$ and $v$ in $G_{A \cup S}$ by the minimality of the separator $S$. Similarly there exists a path $Q$ in $G_{B \cup S}$. We can take $P$ as a shortest path, so that $P$ is chordless. Similarly, $Q$ can be assumed to be a shortest path. The lengths of $P$ and $Q$ are at least two. Then joining two paths $P$ and $Q$ at $u$ and $v$ gives rise to a cycle of length at least 4 which has no chord. This is a contradiction. Hence
There is a simplicial vertex in \( G \).

\( \blacksquare \)

Let \( \text{Theorem 3.5 (Dilac [53])} \)

In a graph \( G \), the \emph{neighborhood} of a vertex \( u \in V(G) \) is the set of vertices adjacent to \( u \), i.e. \( N_G(u) = \{ v \in V(G) : \{ u, v \} \in E(G) \} \). A vertex of a graph is a \emph{simplicial vertex} if its neighborhood is a clique. A \emph{simplicial shelling} is a sequence of deleting a simplicial vertex from a graph. That is, a sequence of vertices \( v_1, v_2, \ldots, v_k \) is a simplicial shelling if \( N_G(v_i) \setminus \{ v_1, \ldots, v_{i-1} \} \) is a clique for every \( i = 1, \ldots, k \). A simplicial shelling is \( v_1, v_2, \ldots, v_m \) a full simplicial shelling if \( \{ v_1, v_2, \ldots, v_m \} = E(G) \). In general a graph has not a full simplicial shelling. This can happen only when a graph is chordal.

**Theorem 3.4 (Dilac [53])** Let \( G \) be a chordal graph. If \( G \) has at least two vertices, it has at least two simplicial vertices. Furthermore, if \( |V(G)| \geq 2 \) and \( G \) is not a clique, there exist at least two non-adjacent simplicial vertices.

\( \text{(Proof)} \) If \( G \) is a complete graph, then the assertion is trivial. Otherwise we use induction on the number \( k \) of the vertices. The case of \( k = 2 \) is trivial. Suppose \( k \geq 3 \). Since \( G \) is supposed to be non-complete, there are two vertices \( a, b \in V(G) \) which are not adjacent. Let \( S \) be a minimal \( a, b \)-vertex separator. Let \( G_A \) and \( G_B \) be the connected component of \( G \setminus V \cup V_S \) including \( a \) and \( b \), respectively. By induction hypothesis, there exists at least two simplicial vertices in \( G_{A \cup S} \), one of which, say \( w \), is not in \( S \) as \( S \) is a clique. Similarly there is a simplicial vertex in \( G_{B \cup S} \) such that \( z \notin S \). Then \( w \) and \( z \) are simplicial vertices of \( G \) that are not adjacent. \( \blacksquare \)

**Theorem 3.5 ([80])** Let \( G \) be a graph. The following are equivalent.

1. \( G \) is a chordal graph.
2. A minimal vertex-separator of \( G \) is necessarily a clique.
3. There exists a simplicial shelling sequence including all the vertices of \( G \).

\( \text{(Proof)} \)

(1) \( \iff \) (2) is already shown in Theorem 3.3.

(1) \( \implies \) (3) By Theorem 3.4, there always exists a simplicial vertex in \( G \). Since deleting a simplicial vertex from a chordal graph leaves a chordal graph again, we can continue deleting a simplicial vertex until the vertex set is empty.

(3) \( \implies \) (1) Let \( C = (v_1, v_2, \ldots, v_k, v_1) \) \( k \geq 4 \) be a cycle. Take a sequence \( (u_1, u_2, \ldots, u_n) \) of a simplicial shelling of \( G \). Suppose \( v_j \) is the vertex in \( C \) which appears first in the sequence \( (u_1, u_2, \ldots, u_n) \). Without loss of generality we can assume \( j = 2 \). Then by definition of simplicial shelling, \( v_1 \) and \( v_3 \) are adjacent, and \( \{ v_1, v_3 \} \) is a chord of \( C \). Hence \( G \) is proved to be a chordal graph. \( \blacksquare \)

**Theorem 3.6 ([73])** Let \( G \) be a chordal graph. Then for any non-simplicial vertex \( v \) of \( G \), there exists a chordless path between two simplicial vertices on which \( v \) lies.
(Proof) For a graph $G = (V, E)$, we use induction on $n = |V|$. For $n = 1$, the assertion is trivial.

Let $v \in V$ be a non-simplicial vertex. Then there exists two vertices $u_1, u_2$ in the neighborhood of $v$ such that they are not adjacent. Suppose $C \subseteq V(G)$ to be a minimal $u_1, u_2$-vertex separator. By definition, obviously $v$ is in $C$. By Theorem 3.5, $C$ is a clique.

$G[V - C]$ has connected components $W_1$ and $W_2$ such that $u_1$ and $u_2$ are included in $W_1$ and $W_2$, respectively. If $u_1$ is a simplicial vertex of $G[W_1 \cup C]$, we set $z_1 = u_1$. Otherwise, by induction hypothesis, $G[W_1 \cup C]$ have two simplicial vertices $y_1, y_2$ such that $u_1$ lies on a chordless path between $y_1$ and $y_2$. Since $C$ is a clique, at least one of $y_1, y_2$ does not belong to $C$. Let $z_1$ be one of $y_1, y_2$ which is not in $C$. Since $G[W_1 \cup \{v\}]$ is connected, there exists a chordless path, say $P_1$, between $z_1$ and $v$ in $G[W_1 \cup \{v\}]$. Similarly $G[W_2 \cup C]$ has a simplicial vertex $z_2$, and there is a chordless path $P_2$ between $v$ and $z_2$ in $G[W_2 \cup C]$.

Joining $P_1$ and $P_2$ at $v$, we have a path $P = P_1 \cup P_2$ such that two end points are simplicial vertices of $G$ and $v$ lies on $P$ as a middle point. Furthermore, $P$ is a chordless path as $C$ is a vertex separator. This completes the induction step. \hfill \Box

3.4 Chromatic polynomials of graphs

Let $G$ be a simple graph, and $[\lambda] = \{1, 2, \ldots, \lambda\}$ for a positive integer $\lambda$. An element of $[\lambda]$ is called a color. A $\lambda$-coloring of a graph $G$ is a map $f : V(G) \rightarrow [\lambda]$. A coloring $f$ is a proper coloring if different colors are assigned to adjacent vertices.

Let $\chi(G; \lambda)$ be the number of the proper coloring of $G$, which we call the chromatic polynomial of $G$. For a complete graph $K_n$, $\chi(K_n; \lambda) = \lambda(\lambda - 1) \cdots (\lambda - (n - 1))$, and for a tree $T$ with $k$ vertices, $\chi(T; \lambda) = \lambda(\lambda - 1)^{k-1}$. By the first sight, it is not sure that $\chi(G; \lambda)$ is a polynomial of $\lambda$. It will be verified from the deletion-contraction rule described below.

Consider the graph in Fig. 3.5, and the edge $e = \{u, v\}$. The chromatic number of $G - e$ is the sum of the two cases: (1) The colors of $u$ and $v$ are different. (2) The colors of $u$ and $v$ are the same. The number of the first case (1) is $\chi(G; \lambda)$, and the number of second case (2) is equal to $\chi(G/e; \lambda)$. Hence it
is intuitively straightforward that \( \chi \) satisfies

\[
\chi(G - e; \lambda) = \chi(G; \lambda) + \chi(G/e; \lambda)
\]  
(3.1)

(3.1) can be restated as

\[
\chi(G; \lambda) = \chi(G - e; \lambda) - \chi(G/e; \lambda)
\]  
(3.2)

(3.1) is called the deletion-contraction rule of a chromatic polynomial. The graphs \( G - e \) and \( G/e \) both have the edges less than those of \( G \). Applying the deletion-contraction rule repeatedly, \( \chi(G; \lambda) \) is decomposed to the sum of the terms of graphs consisting of isolated vertices only. If a graph \( G \) is composed only of \( k \) isolated vertices, \( \chi(G; \lambda) = \lambda^k \). Hence we got to know that \( \chi(G; \lambda) \) is a polynomial of \( \lambda \).

The chromatic polynomial of a chordal graph factors completely into the product of linear terms. It can be deduced from the full simplicial shelling of a chordal. The chromatic polynomial is calculated through backtracking the full simplicial shelling. Consider, for instance, the graph \( G \) in Fig. 3.7. During the backtracking, the number of possible \( \lambda \)-coloring of each new simplicial vertex to be added is firmly determined. Actually, the set \( C \) of vertices adjacent to a new vertex to be added is a clique. Hence the number of possible coloring of the new vertex is equal to \( \lambda - |C| \). Hence the chromatic polynomial of \( G \) is determined to be \( \chi(G; \lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)^2 \).
Chapter 4

Posets and Lattices

4.1 Partial orders

A relation $R$ on a set $S$ is a subfamily $R \subseteq S \times S$. $\leq$ is a partial order on a set $P$ if it satisfies

1. $x \leq x$, (reflexive)
2. if $x \leq y$ and $y \leq x$, then $x = y$, (anti-symmetric)
3. if $x \leq y$ and $y \leq z$, then $x \leq z$. (transitive)

Then $P$ is called a partially ordered set (poset) For $x, y \in P$, $x$ and $y$ are comparable if either $x \leq y$ or $x \geq y$, and otherwise incomparable. We write $x < y$ when $x \leq y$ and $x \neq y$. A subset of a poset is a chain if any two elements in it are comparable, while it is called an antichain if any two elements in it are incomparable. If every pair of the elements of $P$ is comparable, $P$ is called a totally ordered set (or linearly ordered set).

An element is the minimum of $P$, always denoted by $\hat{0}$, if $\hat{0} \leq x$ for every $x \in P$, and an element is the maximum of $P$, denoted by $\hat{1}$, if $x \leq \hat{1}$ for every $x \in P$. Let $(P, \leq)$ be a poset. For any $a, b \in E$ with $a < b$, if $a \leq c \leq b$ implies either $a = c$ or $c = b$, then we say that $b$ covers $a$ or $a$ is covered by $b$, which is denoted by $a \prec b$.

A map $f : P \to P'$ from a poset $(P, \leq)$ to a poset $(P', \leq')$ is order-preserving if $x \leq y$ for $x, y \in P$ implies $f(x) \leq' f(y)$ in $P'$. $f : P \to P'$ is antitone order-preserving if $x \leq y$ for $x, y \in P$ implies $f(y) \leq' f(x)$ in $P'$. An order-preserving map is said to be cover-preserving if $x \prec y$ implies $f(x) \prec f(y)$.

If there exists a bijection $f : P \to P'$ such that $f$ and $f^{-1}$ are both order-preserving maps, it is called an isomorphism between $P$ and $P'$, and $P$ and $P'$ are said to be isomorphic. If $f$ and $f^{-1}$ are antitone order-preserving, then $f$ is called an antitone isomorphism.

For $x, y \in P$ with $x \leq y$, an interval between $x$ and $y$ is the set $[x, y] = \{z \in P : x \leq z \leq y\}$.

The Hasse diagram of a finite poset $P$ is a graph $G$ such that the vertex set $V(G)$ is $P$ and that if $b$ covers $a$ then there is an edge between $a$ and $b$ with $b$ put upper than $a$. Fig. 4.1 is an example of a Hasse diagram. It is immediately seen that $a < e < c$. Also $e$ and $f$ are incomparable.

The following min-max equality is well known.

**Theorem 4.1 (Dilworth’s theorem)** For a finite poset $P$, the minimum number of chains whose union is the entire set $P$ equals to the maximum size of an atichain.

A finite poset has the Jordan-Dedekind chain condition if the lengths of maximal chains connecting any pair of elements are all the same. In addition, if a finite poset $P$ with the minimum element $\hat{0}$ satisfies
the Jordan-Dedekind chain condition, $P$ is called graded. Then the length of the maximal chains between $\hat{0}$ and $x \in P$ is called the height of $x$, denoted by $h(x)$. We call $h$ the height function of $P$.

The dual $\leq^*$ of a partial order $\leq$ is defined by $a \leq^* b \iff b \leq a$ for any $a, b \in E$. 

$(P, \leq^*)$ is said to be the dual poset of $(P, \leq)$.

A subset $I$ of a poset $P$ is an ideal if $x \in I$ and $y \leq x$ imply $y \in I$. An ideal of the form $I(a) = \{x \in P : x \leq a\}$ ($a \in L$) is called a principal ideal. A filter is a subset $F$ such that $y \in F$ and $y \leq x$ entail $x \in F$. The complement of an ideal is a filter, and vice versa. In a word, a filter is the dual notion of an ideal. It is easy to observe that the set of ideals of a poset is closed under set union and intersection.

### 4.2 Lattices

#### Join and Meet

Let $(L, \leq)$ be a poset. An element $z \in L$ is a common upper bound of $x \in L$ and $y \in L$ if $x \leq z$ and $y \leq z$. If there exists a unique minimum element in the common upper bounds of $x$ and $y$, it is called the join of $x$ and $y$, which is denoted by $x \vee y$. Dually, the maximum element of common lower bounds of $x$ and $y$, if it exists, is the meet of $x$ and $y$, which is denoted by $x \wedge y$.

A poset $L$ is a lattice if $x \vee y$ and $x \wedge y$ exist for every $x, y \in L$.

#### Semilattices and Lattices

A poset $L$ is an upper semilattice if $x \vee y$ exists for any $x, y \in L$. A lower semilattice is dually defined.

**Proposition 4.2** A finite upper semilattice containing the minimum element is a lattice with the meet defined by

$$x \wedge y = \bigvee \{z \in L : z \leq x, z \leq y\}. \quad (4.1)$$

Dually, a finite lower semilattice having the maximum element is a lattice.

(Proof) Since the maximum element exists, the right-hand side of (4.1) is not empty, and uniquely determined. It is instant to see that right hand side of 4.1 is really the meet of $x$ and $y$. \qed
In a lattice $L$, an element which covers the minimum element $\hat{0}$ is called an atom. An element covered by the maximum element $\hat{1}$ is called a coatom. If any element $a \in L$ can be represented as the join of atoms, $L$ is atomistic.

A subset $M$ of a lattice $L$ is a sublattice if $M$ is itself a lattice such that the meet and the join of elements in $M$ are equal to those in $L$, i.e., $M$ is closed with respect to the join and the meet of $L$. A sublattice $M$ is said to be cover-preserving if, for $a, b \in M$, $a \prec b$ in $M$ implies $a \prec b$ in $L$. In such a case, we say that $M$ is embedded to $L$.

The fundamental relation between the original partial order and the meet and the join of a lattice is

$$a \leq b \iff a \wedge b = a \iff a \vee b = b.$$  

In a lattice, an element $p \neq \hat{1}$ is join-irreducible if $p = a \lor b$ then either $p = a$ or $p = b$ necessarily holds. An atom is automatically join-irreducible. A meet-irreducible element is dually defined. Clearly, in a finite lattice, any element $x$ other than the minimum $\hat{0}$ is represented as a join of join-irreducible elements. Dually, any element $x$ other than the maximum $\hat{1}$ is a meet of some meet-irreducible elements. In general this representation is not unique. In the later section, we treat with the class of finite lattices such that the set of meet-irreducible elements is unique for the meet-representation of any element.

**Distributive Lattices**

A lattice is distributive if the following distributive law is satisfied.

$$x \wedge (y \lor z) = (x \wedge y) \lor (x \wedge z), \quad x \lor (y \wedge z) = (x \lor y) \wedge (x \lor z) \quad (x, y, z \in L). \quad (4.2)$$

**Proposition 4.3** A finite collection $L$ of sets which is closed under set union and intersection is a distributive lattice with respect to inclusion relation.

(Proof) An inclusion relation on sets is naturally a partial order. If a collection $L$ is closed under union and intersection, the join $A \lor B$ for $A, B \in L$ is $A \cap B$ as well as the meet the join $A \land B$ is $A \cup B$. Hence $L$ is a lattice, and the distributive law (4.2) is satisfied. \[\square\]

As is mentioned later in Theorem 4.9, every finite distributive lattice is isomorphic to a collection of sets which is closed under union and intersection.

A lattice containing the minimum $\hat{0}$ and the maximum $\hat{1}$ is a complemented lattice if for any element $a$, there exits a complemented element $a'$ such that $a \land a' = \hat{0}$, $a \lor a' = \hat{1}$. A complemented distributive lattice is called a Boolean lattice. The power set of a set is naturally a Boolean lattice. Conversely, if a Boolean lattice is finite, it is isomorphic to the power set of a finite set.

**Lemma 4.4** Let $p$ be an element of a distributive lattice. Then

$$p \leq a_1 \lor \cdots \lor a_k \Rightarrow \exists i \ p \leq a_i.$$ 

(Proof) By the distributive law, $p = p \land (\lor_{i=1}^k a_i) = \lor_{i=1}^k (p \land a_i)$. Hence there exists $i$ such that $p = p \land a_i$, namely, $p \leq a_i$. \[\square\]

**Lemma 4.5** Let $L$ be a finite distributive lattice. For every element $a \in L$ except the minimum $\hat{0}$, there exists uniquely a set of join-irreducible elements $p_1, p_2, \ldots, p_k$ such that $a = p_1 \lor p_2 \lor \cdots \lor p_k$. Hence we can define $J(a) = \{p_1, \ldots, p_k\}$ and $J(\hat{0}) = \emptyset$. 

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(Proof) Any element \( a \neq \mathbf{0} \) can be represented by a join of join-irreducible elements. We have to show the uniqueness of this representation. Suppose that \( \{p_1, \ldots, p_k\} \) and \( \{q_1, \ldots, q_m\} \) are the sets of join-irreducible elements and
\[
a = p_1 \lor \cdots \lor p_k = q_1 \lor \cdots \lor q_m.
\]
Then for any \( i \), it follows from Lemma 4.4 that \( p_i \leq q_1 \lor \cdots \lor q_m \). Hence for some \( j \), \( p_i \leq q_j \) holds. Similarly, for some \( h \), we have \( q_j \leq p_h \). Thus \( p_i = q_j \). Repeating this argument leads to \( \{p_i\} = \{q_j\} \). \( \square \)

We remark that \( J(a) \ (a \in L) \) is necessarily an antichain.

Let \( A(L) \) be the set of join-irreducible elements of a finite lattice \( L \). Consider a map \( J : L \to 2^{A(L)} \) such that
\[
J : x \mapsto J(x) = \{a \in A(L) \mid a \leq x\}.
\]
(4.3)

By definition, the following is obvious.

**Lemma 4.6** A map \( J \) is order-preserving.

**Lemma 4.7** For any \( x \in L \), \( x = \bigvee \{a : a \in J(x)\} \).

(Proof) \( x \geq \bigvee_{a \in J(x)} a = x' \) is obvious. Since \( L \) is finite, \( x \) is a join of join-irreducible elements, say, \( b_1, \ldots, b_m \). By definition, \( \{b_1, \ldots, b_m\} \subseteq J(x) \) holds. Hence \( x = b_1 \lor \cdots \lor b_m \leq \bigvee_{a \in J(x)} a = x' \), and \( x = x' \). This completes the proof. \( \square \)

**Lemma 4.8** \( J \) is injective.

(Proof) It is obvious since \( J(x) = J(y) \) implies \( x = y \) from Lemma 4.7. \( \square \)

The collection \( \mathfrak{I}(P) \) of all the ideals of a finite poset \( P \) is closed under set union and set intersection. Hence it is a distributive lattice. A join-irreducible element in the lattice \( \mathfrak{I}(P) \) is a principal ideal.

**Theorem 4.9 (Birkhoff’s theorem)** Let \( L \) be a finite distributive lattice, and \( A(L) \) the set of the join-irreducible elements of \( L \). Also let \( \mathfrak{I}(L) \) be the distributive lattice composed of the ideals of \( A(L) \). Then
\[
\phi : L \to \mathfrak{I}(L), \quad \phi : a \mapsto I(a) = \{p \in L \mid p \leq a\} \quad (a \in L)
\]
is a lattice-isomorphism between \( L \) and \( \mathfrak{I}(L) \).

(Proof) For \( a, b \in L \), if \( a \neq b \), then obviously \( I(a) \neq I(b) \). Hence \( \phi \) is injective.

Take any \( I = \{a_1, \ldots, a_k\} \subseteq \mathfrak{I}(L) \), and put \( b = \bigvee_i a_i \). By definition it is obvious that \( I \subseteq I(b) \).

Conversely, for each \( p \in I(b) \), \( p \leq \bigvee_i a_i \) holds. From Lemma 4.4, \( p \in I \) follows, and we have \( I(b) \subseteq I \). As a result, \( I = I(b) \). Hence every element of \( \mathfrak{I}(L) \) is a principal ideal of \( L \), so that \( \phi \) is bijective. Thus \( \phi \) is surjective. Obviously, \( a \leq b \) in \( L \) if and only if \( I(a) \subseteq I(b) \). Hence \( \phi \) and \( \phi^{-1} \) are both order-preserving. \( \phi \) is actually a lattice-isomorphism. \( \square \)
Modular lattices
A lattice $L$ is modular if for any $x, y, z \in L$,
$$x \leq y \implies (z \lor x) \land y = (z \land y) \lor x.$$ 

We shall define a map between intervals $[a \land b, a]$ and $[b, a \lor b]$,
$$\phi_b : [a \land b, a] \to [b, a \lor b], \quad z \mapsto z \lor b,$$
$$\psi_a : [b, a \lor b] \to [a \land b, a], \quad z \mapsto z \land a.$$ 

**Proposition 4.10** ([20]) A lattice $L$ is modular if and only if for any $a, b \in L$, $\phi_b$ and $\psi_a$ are inverse mappings of each other, and gives a lattice isomorphism between $[a \land b, a]$ and $[b, a \lor b]$.

**Proposition 4.11** ([20]) A lattice is modular if and only if it does not contain $N_5$ as a sublattice.

**Proposition 4.12** ([20]) A lattice is distributive if and only if it contains neither $N_5$ nor $M_3$ as a sublattice.

Given a lattice $L$, the *distributivity kernel* $D_L$ of $L$ is defined as
$$D_L = \{ x \in L : x \land (a \lor b) = (x \land a) \lor (x \land b) \text{ for all } a, b \in L \},$$ (4.4)
and the *modularity kernel* $M_L$ is
$$M_L = \{ x \in L : b \leq x \implies x \land (a \lor b) = (x \land a) \lor b \text{ for all } a, b \in L \}.$$ (4.5)

**Proposition 4.13** A lattice $L$ is distributive if and only if $D_L = L$, and modular if and only if $M_L = L$.

Semimodular Lattices
A finite lattice is an *upper semimodular lattice* (or simply a *semimodular lattice*) if for any $a, b \in L$,
$$a \land b \prec a \implies b \prec a \lor b.$$ 

A *lower semimodular lattice* is the dual of an upper semimodular lattice.

**Lemma 4.14** ([20]) A semimodular lattice satisfies the Jordan-Dedekind chain condition.

**Theorem 4.15** ([20]) Let $L$ be a finite graded lattice with the minimum $\hat{0}$, and $h$ be the height function. Then $L$ is semimodular if and only if for any $x, y \in L$,
$$h(x \land y) + h(x \lor y) \leq h(x) + h(y).$$ 

Furthermore, $L$ is modular if and only if the equality always holds in the above inequalities. In other words, a lattice is modular if and only if it is upper semimodular and lower semimodular at the same time.
Example 4.16 Examples of lattices.

1. An inclusion relation of a family of sets is naturally a partial order. For instance, a family $S_7 = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ of Fig. 4.3 is a poset, and even more a lattice. This lattice is a minimal upper semimodular lattice, but not modular.

2. $L = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}$ is a distributive lattice.

3. Let $E$ be a finite set. An equivalence relation on $E$ has a one-to-one correspondence with a partition of $E$. We can introduce a partial order among all the equivalence relations: For two equivalence relation $\equiv_1$ and $\equiv_2$, $\equiv_1 \leq \equiv_2$ if $a \equiv_1 b$ implies $a \equiv_2 b$ for $a, b \in E$. All the equivalence relations $E$ constitute a lattice with respect to this order. Since a partition of $E$ and an equivalence relation on $E$ is cryptomorphic to each other, the collection of the partitions of $E$ is a lattice isomorphic to it, and called the partition lattice of $E$. When $|E| \geq 3$, a partition lattice is semimodular, but not modular.

4. In the set $\mathbb{N}$ of positive integers, let us denote $a \mid b$ if $a$ divides $b$. With respect to this partial order, $(\mathbb{N}, \mid)$ is an infinite distributive lattice.

Geometric Lattices

An atomistic semimodular lattice is called a geometric lattice. The notion of a geometric lattice is cryptomorphic to a matroid, which will be described in Section 6.1.

If every sublattice of a lattice is complemented, it is said to be relatively complemented.

Proposition 4.17 ([20]) A finite semimodular lattice is geometric if and only if it is relatively complemented.

Inclusion Relation of Classes

Distributive lattices, modular lattices, upper or lower semimodular lattices, geometric lattices, Boolean lattices, etc. are familiar classes of lattices. Among these classes, the main inclusion relation can be described as

$$\text{Boolean} \subset \text{distributive} \subset \text{modular} \subset \text{semimodular}.$$
4.3 Meet-distributive lattices

In [52], Dilworth has proved that for a finite lattice, every element has a unique meet representation if and only if it is an upper semimodular lattice in which every modular sublattice is distributive. This lattice class is rediscovered and named several times independently. This class of lattices is also called upper locally distributive, upper locally free or join-distributive.

The dual lattices of upper locally distributive lattices were called called lower semidistributive lattices by Avann [10]. The class of lower semidistributive lattices has been rediscovered by many others: Boulay (1968); Greene and Markowsky (1974) as lower locally distributive lattices; Jamison (1970, 1980 [94] and 1982 [96]), Edelman (1980 [59]) as meet-distributive lattices. For the historical details, see the note of Monjardet [123].

A finite lattice $L$ is meet-distributive if for every element $x \in L$, the interval $[x_1 \land \ldots \land x_k, x]$ is necessarily a Boolean lattice when $x_1, \ldots, x_k$ are the elements covered by $x$.

We shall give a list of lattice-theoretic characterization of meet-distributive lattices. The join-order function $\tau(x)$ is the number of join-irreducible elements $a \in P$ such that $a \leq x$. For a rank function $r(x)$, $\nu(x) = r(x) - \tau(x)$ is the join-excess function.

**Theorem 4.18 (Avann [10], Bennet [15], Stern [152] p.279, Duquenne [58], [1])** Let $L$ be a finite lattice, and $A(L)$ be the set of join-irreducible elements of $L$. Then the following are equivalent.

1. Every element of $L$ has a unique join decomposition to join-irreducible elements.
2. $L$ is lower semimodular, and every modular sublattice is distributive.
3. $L$ is a lower semimodular lattice which containing no cover-preserving sublattice isomorphic to $M_3$.
   (See Fig. 4.2.)
4. $L$ is a meet-distributive lattice.
5. $L$ is lower-semimodular and join-semidistributive.
6. $L$ is graded with $\nu(\hat{1}) = 0$, namely, $r(\hat{1}) = |A(L)|$.
7. $L$ is graded, and $\nu(x) = 0$ for every $x \in L$.
8. $L$ is a lower semimodular lattice with $\nu(\hat{1}) = 0$.

**Join-semidistributive Lattices**
A lattice is join-semidistributive if for all $x, y, z \in L$,

$$x \lor y = x \lor z \implies x \lor y = x \lor (y \land z).$$

The class of join-semi-distributive lattices is an important extension of the class of distributive lattices. Actually, a lattice is distributive if and only if it is both modular and join-semidistributive. The modular lattice $M_3$ is a typical example that is not join-semidistributive.

For $x, y \in L$, $x \downarrow y$ implies that $x$ is a minimal element of $\{ z \in L : z \not\leq y \}$ and $x \uparrow y$ implies that $y$ is a maximal element of $\{ z \in L : x \not\geq z \}$. $x \downarrow y$ means that $x \downarrow y$ and $x \uparrow y$. We consider the relations $x \downarrow y$, $x \uparrow y$ and $x \downarrow y$ in $A(L) \times M(L)$.

The following characterization is well known.
**Proposition 4.19** Let $L$ be a finite lattice, and $A(L)$ (respectively, $M(L)$) be the set of join-irreducible (respectively, meet-irreducible) elements of $L$. Then $L$ is join-semidistributive if and only if for any $j \in A(L)$, there exists uniquely $m \in M(L)$ such that $j \nmid m$.

**Theorem 4.20** (folklore [1]) A finite lattice $L$ is meet-distributive if and only if it is join-semidistributive and lower semimodular.

**Theorem 4.21** (Adaricheva et al. [1]) Let $L$ be a finite join-irreducible lattice, and $J(L)$ be the set of join-irreducible elements of $L$.

Then $L$ can be embedded to a finite atomistic meet-distributive lattice on the underlying set $J(L)$.

**Corollary 4.22** A finite atomistic join-semidistributive lattice is a finite atomistic meet-distributive lattice.

### 4.4 Incidence algebra and Möbius functions

Let $P$ be a finite poset. The incidence algebra of $P$ is the set of all the real-valued or the integer-valued functions $f(x, y)$ for $x, y \in P$ with the property that $f(x, y) = 0$ if $x \nleq y$ and with the multiplication

$$(f * g)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y). \quad (4.6)$$

The incidence algebra $\mathcal{I}(P)$ of $P$ has the identity element $\delta(x, y)$ such that $\delta(x, y) = 1$ if $x = y$ and $\delta(x, y) = 0$ if $x \neq y$. Actually, $(f * \delta)(x, y) = (\delta * f)(x, y) = f(x, y)$ holds for any $f$.

The zeta function $\zeta(x, y)$ is the element of the incidence algebra of $P$ such that $\zeta(x, y) = 1$ if $x \leq y$ and $\zeta(x, y) = 0$ otherwise. The zeta function has the following properties.

- $\zeta(x, y) = 1$, (for every interval $[x, y]$)
- $\zeta^2(x, y) = \sum_{x \leq z \leq y} \zeta(x, z)\zeta(z, y) = \sum_{x \leq z \leq y} 1 = \text{the cardinality of the interval } [x, y],$
- $\zeta^3(x, y) = \sum_{x \leq z \leq y} \zeta(x, z)\zeta(z, y) = \sum_{x \leq z \leq y} \zeta^2(z, y) = \sum_{x \leq z \leq y} |[z, y]|,$
- $\zeta^k(x, y)$ = the number of chains $x = x_0 \leq x_1 \leq \cdots \leq x_k = y$ with repetitions.

When a finite poset $P$ has the minimum $\hat{0}$ and the maximum $\hat{1}$, the function

$$Z(P, n) = \zeta^n(\hat{0}, \hat{1}) = \sharp \{ \hat{0}-\hat{1} \text{ chains of length } n \text{ with repetitions } \} \quad (n \text{ is a positive integer})$$

is a polynomial in the variable $n$, called the zeta polynomial.

The Möbius function of $P$ is defined as follows. The Möbius function $\mu : P \times P \to \mathbb{Z}$ is an integer-valued function defined recursively by

$$\mu(x, x) = 1, \quad (x \in P)$$
$$\mu(x, y) = 0, \quad (x \nleq y)$$
$$\mu(x, y) = -\sum_{x \leq z \leq y} \mu(x, z). \quad (x < y) \quad (4.7)$$
Theorem 4.23 (Rota [144]) The zeta function of a finite set $P$ has its inverse. Actually, the Möbius function $\mu(x, y)$ defined in (4.7) is the inverse of the zeta function, i.e. $(\mu \ast \zeta)(x, y) = (\zeta \ast \mu)(x, y) = \delta(x, y)$.

The inversion formula below is the essential property of the Möbius function.

Theorem 4.24 (Möbius Inversion Theorem [2, 144]) Let $P$ be a finite poset. Let $f, g$ be integer-valued functions from $P$ or real-valued functions from $P$. Then

1. Inversion from above:
   \[ g(x) = \sum_{x \leq y} f(y) \quad (x \in P) \iff f(x) = \sum_{x \leq y} \mu(x, y)g(y) \quad (x \in P) \]

2. Inversion from below:
   \[ g(x) = \sum_{y \leq x} f(y) \quad (x \in P) \iff f(x) = \sum_{y \leq x} g(y)\mu(y, x) \quad (x \in P) \]

(Proof) The proof of (1):
\[
\sum_{x \leq y} \mu(x, y)g(y) = \sum_{x \leq y} \mu(x, y) \left( \sum_{y \leq z} f(z) \right) = \sum_{x \leq y} \mu(x, y) \left( \sum_{y \leq z} f(z)\zeta(y, z) \right)
\]
\[
= \sum_{x \leq y} \left( \sum_{y \leq z} f(z)\mu(x, y)\zeta(y, z) \right) = \sum_{x \leq z} \left( f(z) \sum_{x \leq y \leq z} \mu(x, y)\zeta(y, z) \right)
\]
\[
= \sum_{x \leq z} f(z)\delta(x, z) = f(x)
\]

The proof of (2) is similar. □

We show a simple application of Theorem 4.24. Let $2^E$ be the Boolean lattice of subsets of a nonempty finite set $E$. The Möbius function $\mu$ of $2^E$ is easy to calculate: For $X, Y \subseteq E$,

\[
\mu(X, Y) = \begin{cases} 
(-1)^{|Y| - |X|} & \text{if } X \subseteq Y, \\
0 & \text{otherwise.}
\end{cases}
\]

Hence

Corollary 4.25 Let $f : 2^E \to \mathbb{R}$ be a real-valued function. Define $g : 2^E \to \mathbb{R}$ by

\[ g(Y) = \sum_{Y \subseteq X} f(X) \quad \text{for } Y \subseteq E. \]

Then

\[ f(X) = \sum_{X \subseteq Y} (-1)^{|Y| - |X|}g(Y) \quad \text{for } X \subseteq E. \]

Example 4.26 Let $\{A_i : i \in I\}$ be a collection of subsets of a nonempty finite set $E$. We can deduce the following Inclusion-Exclusion formula from the inversion formula of Theorem 4.24.

\[ \left| \bigcup_{i \in I} A_i \right| = \sum_{X \subseteq I} (-1)^{|X|-1} \left| \bigcap_{i \in X} A_i \right|. \]
4.5 Characteristic functions and $\beta$-invariants

Let $P$ a finite graded poset with the minimum $\hat{0}$ and the maximum $\hat{1}$. Let $h$ be the height function of $P$. Based on a Möbius function, a characteristic function $p(P; \lambda)$ and a $\beta$-invariant of the poset $P$ are defined as follows. The characteristic function has the historical origin in the chromatic polynomials of graphs.

$$p(P; \lambda) = \sum_{x \in P} \mu(\hat{0}, x) \lambda^{h(x)} - h(x), \quad (4.8)$$

$$\beta(P) = -\frac{d}{d\lambda} \left|_{\lambda=1} p(P; \lambda) \right| = \sum_{x \in P} \mu(\hat{0}, x) h(x). \quad (4.9)$$

4.6 Supersolvable lattices

Let $L$ be a finite lattice. $L$ is supersolvable if there exists a maximal chain $D$ such that a sublattice generated by any other chain and $D$ is always distributive (See [27, 149]). $D$ is called an M-chain of $L$. By definition, the dual of a supersolvable lattice is supersolvable as well. A supersolvable lattice is known to satisfy the Jordan-Dedekind chain condition. The terminology comes from the lattice of subgroups of a supersolvable group.

Theorem 4.27 (Stanley [149]) Let $L$ be an upper semimodular supersolvable lattice of rank $n$ with an $M$-chain $\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$, and let $a_i$ be the number of atoms $x$ of $L$ satisfying $x \leq x_i$ and $x \not\leq x_{i-1}$. Then the characteristic polynomial of $L$ factors completely as follows.

$$p(L; \lambda) = (\lambda - a_1)(\lambda - a_2) \cdots (\lambda - a_n). \quad (4.10)$$

Furthermore, the multi-set $\{a_1, \ldots, a_n\}$ does not depend on the choice of an M-chain.

The collection of flats of a matroid forms an atomistic upper semimodular lattice, called a geometric lattice. If the geometric lattice of a matroid is supersolvable, we call it a supersolvable matroid. For a supersolvable matroid, the characteristic polynomial factors completely as in Theorem 4.27. The characterizations of supersolvable matroids will be presented as Theorem 6.20 in Section 6.6.

Suppose $P$ to be a finite graded poset with minimum $\hat{0}$ and the maximum $\hat{1}$. Let $h : P \to \mathbb{Z}_+$ be the height function, and suppose $h(\hat{1}) = n$. Let $E(P) = \{(x, y) : y \text{ covers } x \text{ in } P\}$, i.e. $E(P)$ be the edge set of the Hasse diagram of $P$. A function $f : E(P) \to [n] = \{1, 2, \ldots, n\}$ is called an edge-labelling. For an interval $[a, b]$ in $P$ and each maximal chain $\beta : a = a_0 < a_1 < \cdots < a_k = b$ in $[a, b]$, we define a sequence $s(\beta) = (f(a_0, a_1), f(a_1, a_2), \ldots, f(a_{k-1}, a_k))$. The chain $\beta$ is said to be increasing if $f(a_0, a_1) \leq f(a_1, a_2) \leq \cdots \leq f(a_{k-1}, a_k)$.

An edge-labelling $f$ is called an $S_n$ EL-labeling if the following hold:

1. Every interval $[a, b]$ in $P$ has exactly one increasing maximal chain.
2. For every maximal chain $\beta$ in an interval $[a, b]$ in $P$, $s(\beta)$ is a permutation of a fixed subset of $\{1, 2, \ldots, n\}$.

Hence for an $S_n$ EL-labelling $f$, there exists uniquely a maximal increasing chain:

$$\hat{0} = a_0 < a_1 < \cdots < a_n = \hat{1} \quad (4.11)$$
If a finite graded lattice $L$ is supersolvable with an M-chain (4.11), we can define an $S_n$ EL-labelling $f$ by

$$f(x, y) = \min\{i : y \leq a_i \lor x\}$$

for each covering pair $x \prec y$ in $L$.

**Theorem 4.28 (McNamara [121])** A finite graded lattice of height $n$ is supersolvable if and only if it has an $S_n$ EL-labelling. In particular, the unique maximal increasing chain with this labelling is an M-chain.

### 4.7 Convex dimension of lattices

Let $L$ be a finite lattice. For a collection of chains $C_1, \ldots, C_k$ in $L$, their intersection is $\{x_1 \land \cdots \land x_k : x_i \in C_i \text{ for } i = 1, \ldots, k\}$. Their union is dually defined.

The *convex dimension* $\text{cdim}(L)$ of a finite lattice $L$ is the minimum size of maximal chains in $L$ whose intersection gives rise to $L$. The *concave dimension* $\text{cadim}(L)$ is the minimum size of maximal chains of $L$ whose union gives $L$.

The *width* $w(Q)$ of a poset $Q$ is the maximum cardinality of antichains in $Q$.

**Theorem 4.29 ([67])** Let $L$ be a finite lattice. The convex dimension of $L$ is equal to the width of the set of meet-irreducible elements of $L$. Dually, the concave dimension of $L$ is the width of the set of join-irreducible elements of $L$.

(Proof) Let $M(L)$ be the set of meet-irreducible elements of $L$. Let $C_1, \ldots, C_k$ be chains in $L$ such that $\bigwedge_{i=1}^k C_i = \{x_1 \land \cdots \land x_k : x_i \in C_i \text{ for } 1 \leq i \leq k\} = L$ and $k$ is the smallest number with respect to this property. Then $M(L) \subseteq C_1 \cup \cdots \cup C_k$ since any meet-irreducible elements cannot be a meet of distinct elements. Hence $w(L) \leq k$.

As easily seen, $\{M(L) \cap C_i : i = 1, \ldots, k\}$ is a partition of $M(L)$. By Theorem 4.1, $k \leq w(M(L))$. This completes the proof of the first half. The latter part can be shown in a dual way. □

The collection of all the lattices on a fixed underlying set $E$ constitutes a meet-semilattice $S_E$ with respect to an inclusion relation of the relations on $E$. An element of this meet-semilattice is maximal if and only if it is a total ordering. The *dimension* $\text{dim}(S_E)$ of the semilattice $S_E$ is the smallest number of maximal chains whose intersection equals to $S_E$.

**Proposition 4.30 (Kelly and Trotter [106])** For any finite lattice $L$, $w(M(L)) \geq \text{dim}(L)$ and $w(J(L)) \geq \text{dim}(L)$. 

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Chapter 5

Closure Systems

5.1 Closure systems

Let \( E \) be a nonempty finite set. \( \mathcal{K} \subseteq 2^E \) is a closure system or Moore family if it satisfies the following.

1. \( E \in \mathcal{K} \),
2. \( X, Y \in \mathcal{K} \implies X \cap Y \in \mathcal{K} \).

We call an element of \( \mathcal{K} \) a closed set. The complement of a closed set is an open set. The collection of all the open sets, namely the complement of \( \mathcal{K} \), is an open-set family. Since \( \mathcal{K} \) is a finite lower semilattice with respect to set intersection having the maximum element \( E \), by Proposition 4.2, it is actually a lattice.

A map \( \sigma : 2^E \to 2^E \) is a closure operator if it satisfies

1. \( A \subseteq \sigma(A) \), (extensive)
2. If \( A \subseteq B \) then \( \sigma(A) \subseteq \sigma(B) \), (monotone)
3. \( \sigma(\sigma(A)) = \sigma(A) \). (idempotent)

The notions of closure systems and closure operators are equivalent. For a closure system \( \mathcal{K} \), \( \sigma : 2^E \to 2^E \) below is a closure operator.

\[
\sigma(A) = \bigcap_{X \in \mathcal{K}, A \subseteq X} X.
\] (5.1)

Conversely, \( \mathcal{K} = \{ X \subseteq E : \sigma(X) = X \} \) holds. Hence there is a one-to-one correspondence between the closure systems and the closure operators. We shall call the corresponding triple \( (\mathcal{K}, \sigma, E) \) a closure space where \( \mathcal{K} \) or \( \sigma \) is redundant. So we shall also call \( (\mathcal{K}, E) \) or \( (\sigma, E) \) together with the associated closure operator or closure system implicitly.

We can easily generalize a closure operator to a poset. For a poset \( P \), a map \( \varphi : P \to P \) is a closure operator if it satisfies

1. \( a \leq \varphi(a) \), \( (a \in P) \)
2. If \( a \leq b \), then \( \varphi(a) \leq \varphi(b) \), \( (a, b \in P) \)
3. \( \varphi(\varphi(a)) = \varphi(a) \). \( (a \in P) \)
We shall explain later that the collection of flats of a matroid constitutes a closure system. A matroid can be characterized as a closure system whose closure operator satisfies the exchange property, while a convex geometry is a closure system whose closure operator meets the anti-exchange property. Under this exposition, a closure system is a common generalization of a matroid and a convex geometry. Hence we emphasize that the notion of closure system gives a unified point of view to the theory of matroids and convex geometries both.

**Proposition 5.1** For a closure space \((\mathcal{K}, \sigma, E)\), if \(\sigma(A) \cup \sigma(B) = \sigma(A \cup B)\) for any \(A, B \subseteq E\), then \(\mathcal{K}\) is a distributive lattice.

(Proof) By assumption, \(\mathcal{K} = \{\sigma(A) : A \subseteq E\}\) is closed under set union and intersection. Hence the assertion immediately follows from Proposition 4.3. \(\square\)

Note that the converse is not true. That is, even if a closure system is a distributive lattice, it is not necessarily closed with respect to set union in general.

A map \(\text{ex} : 2^E \to 2^E\)

\[
\text{ex}(A) = \{x \in A : x \notin \sigma(A - x)\} \quad (A \subseteq E)
\]

is called the *extreme-point operator* of a closure system. An element in \(\text{ex}(A)\) is called an *extreme element* of \(A\).

If \(\text{ex}(A) = A\), \(A\) is said to be an *independent set*. That is, \(A\) is an independent set if and only if for each \(e \in A\) \(e \notin \sigma(A - e)\) holds. By definition, the empty set is necessarily an independent set. By Proposition 5.2 below, the collection of the independent sets is an independence system. A set which is not independent is *dependent*, and a minimal dependent set is called a *circuit*. In the case of matroid theory, which will be described in Chapter 6, the ordinary definitions of independent set and circuit agree with those shown above.

We shall define a coloop and a loop of a closure system.

(a) An element in \(\text{ex}(E)\) is a *coloop* of \((\mathcal{K}, E)\).

(b) An element in \(\sigma(\emptyset) = \bigcap_{X \in \mathcal{K}} X\) is a *loop*. (Note that \(\sigma(\emptyset)\) is the minimum element of \(\mathcal{K}\).)

If a closure system contains no loop, or equivalently, if \(\emptyset \in \mathcal{K}\), then the closure system is called *loop-free*. A closure system is *atomistic* if \(\sigma(\{a\}) = \{a\}\) for every \(a \in E\). An atomistic closure system is necessarily loop-free, but the converse is not true in general.

**Proposition 5.2** In a closure system, a subset of an independent set is again an independent set. Hence the collection of the independent sets of a closure system forms a simplicial complex, denoted by \(\text{In}(\mathcal{K})\).

(Proof) Assume \(A \subseteq E\) to be an independent set, and take any subset \(A' \subseteq A\). By assumption, for any \(e' \in A'\), we have \(e' \notin \sigma(A - e')\). In the meanwhile, \(\sigma(A' - e') \subseteq \sigma(A - e')\) follows from the monotonicity. Hence we have \(e' \notin \sigma(A' - e')\). Since \(e'\) is arbitrarily chosen, \(\text{ex}(A') = A'\) holds, which implies that \(A'\) is an independent set. \(\square\)

If \(A\) is both independent and closed, it is said to be a *free set*.

If a closure system contains a loop, then there does not exist a free set. In fact, if \(p\) is a loop in a closure system, then \(p\) must belong to every closed set. In the meanwhile, for any closed set \(X\), \(p \notin \text{ex}(X)\)
holds. This implies that there is no closed set \( X \) with \( \text{ex}(X) = X \). Henceforth, in case that a free set is under consideration, we can suppose that a closure system is loop-free.

**Theorem 5.3** In a loop-free closure system, a subset \( A \subseteq E \) is a free set if and only if every subset of \( A \) is a closed set.

(Proof) Suppose first that \( A \) is a free set. Take any subset \( B \subseteq A \), and we shall show that \( B \) is closed. Suppose contrarily that \( B \) is not closed. Then, by definition, there exists an element \( e \in \sigma(B) - B \). Since \( e \notin B \subseteq A - e \) is trivial, \( e \in \sigma(B) \subseteq \sigma(A - e) \) immediately follows. Apart from this, by the definition of independence, we have \( e \in A = \text{ex}(A) \), which implies \( e \notin \sigma(A - e) \). This is a contradiction.

Conversely suppose that every subset of \( A \) is closed. Hence for any element \( e \in A \), \( A - e \) is a closed set. Then \( \text{ex}(A) = \{ e \in A : e \notin \sigma(A - a) \} = \{ e \in A : e \notin A - a \} = A \). Now \( A \) is proved to be an independent set, and the assertion holds. \( \square \)

**Corollary 5.4** Any subset of a free set is a free set again.

**Corollary 5.5** In a loop-free convex geometry, a necessary and sufficient condition for a closed set \( X \in K \) to be free is that the interval \( [\emptyset, X]_K \) in the lattice \( K \) forms a Boolean lattice.

Corollary 5.4 implies that the collection of free sets forms a simplicial complex, which we denote \( \text{Free}(K) \). Since a free set is independent, \( \text{Free}(K) \) is a subcomplex of \( \text{In}(K) \).

### 5.2 Minors of closure systems

We shall state the deletion, the contraction, and the trace of Section 2.1 once again for closure systems. Let \((K, O)\) be a complement pair of a closure system \( K \) and an open-set system \( O \) on \( E \).

(A) **Trace**: For any \( A \subseteq E \),

\[
K - A = \{ X \setminus A : X \in K \},
\]
\[
O - A = \{ X \setminus A : X \in O \},
\]

are a complement pair of a closure system and an open-set system on \( A \). These are called *trace minors*. They are also denoted as \( K : X = K - (E \setminus X) \) and \( O : X = O - (E \setminus X) \). \((K - A)^* = O - A \) obviously holds.

(B) **Contraction of \( K \) and deletion of \( O \)**: For any \( A \subseteq E \),

\[
K /A = \{ X - A : X \in K, A \subseteq X \},
\]
\[
O \setminus A = \{ X : X \in O, X \cap A = \emptyset \}.
\]

\((K /A)^C = O \setminus A \) is obvious. \( K /A \) is called the *contraction* of \( K \) by \( A \).

(C) **Deletion of \( K \) and contraction of \( O \)**: Only in case that \( E - A \in K \) and \( A \in O \), we consider the complement pair

\[
K \setminus A = \{ X : X \in K, X \subseteq E - A \},
\]
\[
O /A = \{ X - A : X \in O, A \subseteq X \},
\]

since otherwise \( K \setminus A \) ( \( O /A \), respectively ) may not be a closure system (an open-set system, respectively). \( K \setminus A \) is called the *deletion* of \( K \) by \( A \).
Proposition 5.6 For a closure space \((K, E)\) and a subset \(A\) of \(E\), if \(E - A \in K\), then \(K \setminus A = K - A\).

(Proof) Since \(E - A \in K\), for any \(X' \in K\), \(X' \cap (E - A) \in K\) holds. Hence for \(X \subseteq E - A\),

\[
X \in K - A \iff X = X' \setminus A \text{ for } \exists X' \in K \iff X = X' \cap (E - A) \text{ for } \exists X' \in K
\]

\[
\iff X \subseteq E - A, \ X \in K \iff X \in K \setminus A.
\]

This completes the proof. \(\Box\)

(D) We write \(K\lfloor F\) to denote \(K \setminus (E - F)\). For \(F, G \in K\) with \(F \subseteq G\), \([F, G] = \{X \in K : F \subseteq X \subseteq G\}\) is an interval of \(K\). We shall call \((K\lfloor G)\)/\(F = \{X - F : X \in [F, G]\}\) an interval minor, which is isomorphic to the interval \([F, G]\) as a lattice. An interval minor of an open-set system is defined in a dual way.

Proposition 5.7 Let \((K, \sigma)\) be a closure space on \(E\), and \(A \subseteq E\).

(1) The closure operator \(\sigma^A\) of the trace \(K - A\) is

\[
X \mapsto \sigma^A(X) = \sigma(X) \setminus A \quad \text{for } X \subseteq E - A. \tag{5.3}
\]

(2) The closure operator \(\sigma_A\) of the contraction \(K/A\) is

\[
X \mapsto \sigma_A(X) = \sigma(X \cup A) \setminus A \quad \text{for } X \subseteq E - A. \tag{5.4}
\]

(3) Suppose \(E - A \in K\). The closure operator \(\tilde{\sigma}\) of the deletion \(K \setminus A\) is

\[
X \mapsto \tilde{\sigma}(X) = \sigma^A(X) = \sigma(X) \quad \text{for } X \subseteq E - A. \tag{5.5}
\]

(Proof) We prove (1). Since \(X \cap A = \emptyset\), observe that for any \(Y \subseteq E\), \(X \subseteq Y \setminus A\) if and only if \(X \subseteq Y\).

\[
\sigma(X) \setminus A = \left( \bigcap_{X \subseteq Y, Y \in K} Y \setminus A \right) = \bigcap_{X \subseteq Y, Y \in K} (Y \setminus A) = \bigcap_{X \subseteq Y', Y' \in K - A} Y' = \sigma^A(X).
\]

Next (2) is to be proved. Recall the observation of the proof of (1).

\[
\sigma_A(X) = \bigcap_{X \subseteq Y, Y \in K/A} Y = \bigcap_{X \subseteq Y' - A, A \subseteq Y', Y' \in K} (Y' - A) = \left( \bigcap_{X \subseteq Y', A \subseteq Y', Y' \in K} Y' \right) \setminus A
\]

\[
= \left( \bigcap_{X \cup A \subseteq Y', Y' \in K} Y' \right) \setminus A = \sigma(X \cup A) \setminus A.
\]

Lastly we prove (3). For \(X \subseteq E - A \in K\), we have \(\sigma(X) \subseteq E - A\) and \(\sigma(X) \cap A = \emptyset\). By Proposition 5.6, \(\tilde{\sigma}(X) = \sigma^A(X)\). Hence \(\tilde{\sigma}(X) = \sigma^A(X) = \sigma(X)\). This completes the proof. \(\Box\)
5.3 Lattices as closure systems

First recall that a closure system forms a lattice. Conversely we shall show that any finite lattice is isomorphic to a certain closure system. For a lattice $L$, $A(L)$ denotes the set of join-irreducible elements of $L$. Let $J : L \to 2^{A(L)}$ be the map defined in (4.3).

Lemma 5.8 For a finite lattice $L$, $\mathcal{K} = \{ J(x) : x \in L \}$ is a closure system.

(Proof) By definition, it is clear that $\mathcal{K}$ contains $A(L) = J(1)$, and that $J(x) \cap J(y) = J(x \land y)$ since for $a \in A(L)$, $a \in J(x) \cap J(y) \iff a \in J(x)$ and $a \in J(y) \iff a \leq x$ and $a \leq y \iff a \leq x \land y$. Hence it is a closure system. □

Proposition 5.9 A finite lattice $L$ is isomorphic to $\mathcal{K} = \{ J(x) : x \in L \}$. In particular, every finite lattice is isomorphic to the lattice of a closure system.

(Proof) By definition, $J$ is surjective. From Lemmas 4.6 and 4.8, $J$ is bijective and order-preserving. Hence it is an isomorphism. The latter part follows from Lemma 5.8. □

Example 5.10 The set of join-irreducible elements of the lattice of Fig.5.1 is $\{a, b, c, d, e\}$. Fig.5.2 depicts the closure system constructed as is indicated in Proposition 5.9, and hence is isomorphic to the lattice Fig. 5.1.

5.4 Rooted circuits of closure systems

First recall that a rooted set is a pair of a subset $X$ and an element $e$ with $e \not\in X$ in our terminology. Any collection of rooted sets gives rise to a closure system. The converse will be proved in Theorem 5.15.

Proposition 5.11 Let $\mathcal{R} = \{(X_i, e_i) : i = 1, \ldots, m\}$ ($X_i \subseteq E, e_i \in E$) be a collection of rooted sets on $E$. Then

$$\mathcal{K}[\mathcal{R}] = \{ A \subseteq E : \text{for every } (X, e) \in \mathcal{R}, \text{ if } X \subseteq A, \text{ then } e \in A \}$$

$$= \{ A \subseteq E : \text{for every } (X, e) \in \mathcal{R}, \text{ either } X \not\subseteq A \text{ or } e \in A \}$$

is a closure system.
Figure 5.3: A closure system

(Proof) Let $A, B \in \mathcal{K}[\mathcal{R}]$ and take any $(X, e) \in \mathcal{R}$. If either $X \not\subseteq A$ or $X \not\subseteq B$, then clearly $X \not\subseteq A \cap B$, hence $A \cap B \in \mathcal{K}[\mathcal{R}]$. The last case to check is that $e \in A$ and $e \in B$ both hold. Then $e \in A \cap B$ is obvious, and $A \cap B \in \mathcal{K}[\mathcal{R}]$. □

For a closure space $(\mathcal{K}, \sigma, E)$, a rooted set $(X, e)$ is a rooted circuit of $\mathcal{K}$ if $X$ is a minimal set in $E \setminus e$ such that $e \in \sigma(X)$. For a rooted circuit $(X, e)$, $X$ is called its stem, and $e$ is its root [112]. Note that in the former literatures such as [50, 112] they use a different notation, and call $(X \cup e, e)$ a rooted circuit.

Let $\mathcal{C}(\mathcal{K})$ denote the collection of all the rooted circuits of $\mathcal{K}$. $\mathcal{C}(e)$ denotes the collection of all the stems of a fixed root $e$, that is $\mathcal{C}(e) = \{X : (X, e) \in \mathcal{C}(\mathcal{K})\}$. By definition, $\mathcal{C}(e)$ is necessarily a clutter.

A rooted circuit $(X, e)$ is said to be a critical rooted circuit if $\sigma(X)(= \sigma(X \cup e))$ is minimal among the closures of rooted circuits such that $e$ is their root.

Example 5.12 Fig. 5.3 shows a closure system on $E = \{a, b, c, d\}$. It has four rooted circuits: $(\{a, c\}, b)$ and $(\{b, d\}, c)$ are critical rooted circuits whereas $(\{a, d\}, b)$ and $(\{a, d\}, c)$ are non-critical rooted circuits.

Recall that a circuit (i.e. a minimal dependent set) of a closure system includes at least one non-extreme element. A circuit of a closure system corresponds to a rooted circuit, but in general the converse is not true.

Theorem 5.13 Let $(\mathcal{K}, E)$ be a closure system, $e$ be an element of $E$, and $X$ be a subset of $E - e$. If $C = X \cup e$ is a circuit and $e$ is its non-extreme element, then $(X, e)$ is a rooted circuit of $\mathcal{K}$.

(Proof) First we shall show $e \in \sigma(X)$. By the choice of $e$, $e \not\in \text{ex}(C) = \text{ex}(X \cup e)$, which is equivalent to $e \in \sigma(X)$. Next we shall show the minimality of $X$. Suppose that there exists $X' \subseteq X$ and $e \in \sigma(X')$. Then $X' \cup e$ is a dependent set, and contains a circuit, which contradicts the minimality of a circuit $C$. This completes the proof. □

In case of a matroid, for any circuit $C$ and any element $e \in C$, $e$ is a non-extreme element of $C$. $(C \setminus e, e)$ is a rooted circuit, and an arbitrary rooted circuit arise in this way. In a convex geometry, a circuit $C$ has a unique non-extreme element $e \in C$, and $(C \setminus e)$ is a rooted circuit, which will be explained in Theorem 7.11.
Example 5.14 The converse of Theorem 5.13 is not true in general for closure systems. A closure system \( K = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, b, c\}\) is a counterexample for it. \( \{\{a, c\}, d\} \) is a rooted circuit, but \( \{a, c, d\} \) is not a circuit (i.e. a minimal dependent set).

The collection of rooted circuits of a closure system conversely determines the closure system and the closure operators.

Theorem 5.15 Let \( C \) be the collection of rooted circuits of a closure system \( K \). Then \( K \) is given by \( C \) as \( K[C] = K \). Namely,

\[
K = \{A \subseteq E : \text{for every } (X, e) \in C, \text{if } X \subseteq A, \text{then } e \in A\}
\]

(Proof) Suppose that \( A \in K \) is not in \( K[C] \). Then there exists a rooted circuit \( (X, e) \in C \) such that \( e \not\in A \) and \( X \subseteq A \). But by definition, \( e \in \sigma(X) \subseteq \sigma(A) = A \) holds, a contradiction.

Conversely, suppose that \( A \in K[C] \) is not in \( K \). By assumption \( A \subseteq \sigma(A) = A' \). Take any \( e \in A' - A \).

Since \( \sigma(A) = A' \), there exists a set \( C \) such that \( C \subseteq A, e \in \sigma(C) \). Take such a \( C \) that \( C - e = X_0 \) is minimal. By definition, \( (X_0, e) \) is a rooted circuit, and \( e \in X_0 \) and \( X_0 \subseteq A \), which implies \( A \not\in K[C] \), a contradiction. \( \square \)

Theorem 5.16 For an arbitrary closure operator \( \sigma \), it holds that

\[
\sigma(A) = A \cup \{e \in E - A : \exists X \subseteq A, (X, e) \text{ is a rooted circuit}\} \quad (A \subseteq E)
\]

(Proof) For any \( a \in E - A \), if \( a \in \sigma(A) \), by definition there exists a subset \( X \subseteq A \) such that \( (X, a) \) is a rooted circuit. Since \( a \in \sigma(X) \subseteq \sigma(A) \), the converse is obvious. \( \square \)

5.5 Connectivity of closure systems

Suppose that \( (K_1, \sigma_1, E_1) \) and \( (K_2, \sigma_2, E_2) \) are closure spaces, and \( E_1 \cap E_2 = \emptyset \). Then

\[
K = K_1 \oplus K_2 = \{X \cup X' : X \in K_1, X' \in K_2\} \quad (5.6)
\]

is clearly a closure system on \( E = E_1 \cup E_2 \), which we call the direct sum of \( K_1 \) and \( K_2 \). Note that \( E_1 \) is not a closed set in \( K \) unless \( \emptyset \in K_2 \), and similarly \( E_2 \) is not closed in \( K \) unless \( \emptyset \in K_1 \). It is obvious that

\[
K_1 = K[E_1] = \{X \cap E_1 : X \in K\}, \quad K_2 = K[E_2] = \{X \cap E_2 : X \in K\}
\]

Let \( \sigma \) be a closure operator of \( K \). Then \( \sigma(A) = \sigma_1(A \cap E_1) \cup \sigma_2(A \cap E_2) \) \( (A \subseteq E) \).

If there is a partition \( \{E_1, E_2\} \) of \( E \) into nonempty subsets and \( K = (K[E_1]) \oplus (K[E_2]) \), then \( K \) is disconnected. A closure system is connected unless it is disconnected.

For a closure system \( K \), \( Z = \cap\{X : X \in K\} \) is the set of its loops. If \( Z = \emptyset \), then \( K \) is said to be loop-free. If \( Z \) is nonempty, we have a trivial direct-sum decomposition \( K = (K : (E \setminus Z)) \oplus 2^Z \). Hence if a closure system with \(|E| \geq 2\) contains a loop, it is necessarily disconnected.

A connected closure system is defined to be 2-separable if \( K - e \) is disconnected for some \( e \in E \). Otherwise it is 2-connected.

Lemma 5.17 Suppose \( K \) to be a loop-free closure system decomposed into a direct sum as \( K = (K[E_1]) \oplus (K[E_2]) \). Then \( \emptyset \in K[E_1] \) and \( \emptyset \in K[E_2] \). At the same time \( E_1 \) and \( E_2 \) are both closed sets of \( K \).
Theorem 5.18 Let $(\mathcal{K}, \sigma, E)$ be a loop-free closure space. Then for a partition $\{E_1, E_2\}$ of $E$, the following statements are equivalent.

(1) $\mathcal{K} = (\mathcal{K}|E_1) \oplus (\mathcal{K}|E_2)$.

(2) For each $A \subseteq E$, putting $A_1 = A \cap E$ and $A_2 = A \cap E$, it holds that $\sigma(A_1) \subseteq E_1$, $\sigma(A_2) \subseteq E_2$, and $\sigma(A) = \sigma(A_1) \cup \sigma(A_2)$.

(Proof)

(1) $\Rightarrow$ (2) From Lemma 5.17 and the assumption, $E_1$ and $E_2$ are closed sets in $(E, \mathcal{K})$, which implies $\sigma(A_1) \subseteq E_1$ and $\sigma(A_2) \subseteq E_2$. It is also obvious that

$$A = A_1 \cup A_2 \subseteq \sigma(A_1) \cup \sigma(A_2) \subseteq \sigma(A_1 \cup A_2) = \sigma(A).$$

Now it follows from the assumption (1) that $\sigma(A_1) \cup \sigma(A_2)$ is a closed set of $(E, \mathcal{K})$. Hence,

$$\sigma(A) \subseteq \sigma(\sigma(A_1) \cup \sigma(A_2)) = \sigma(A_1) \cup \sigma(A_2) \subseteq (A).$$

Thus we have $\sigma(A) = \sigma(A_1) \cup \sigma(A_2)$.

(2) $\Rightarrow$ (1) It is sufficient to show that for any $X \in \mathcal{K}|E_1$ and $Y \in \mathcal{K}|E_2$, $X \cup Y \in \mathcal{K}$ is attained. From the assumption, we have $\sigma(E_1) = E_1$ and $\sigma(E_2) = E_2$, and hence $E_1$ and $E_2$ are both closed sets of $\mathcal{K}$. Accordingly, $X$ and $Y$ are closed sets of $\mathcal{K}$ as well. It directly follows from (2) that $\sigma(X \cup Y) = \sigma(X) \cup \sigma(Y) = X \cup Y$. Hence we have $X \cup Y \in \mathcal{K}$. \hfill $\square$

We define an undirected graph $G = (E, A)$ from the set $\mathcal{C}$ of rooted circuits of a closure system $(\mathcal{K}, E)$. The vertex set is $E$, and the edge set is $A = \{\{e, f\} \mid \exists (X, e) \in \mathcal{C}, f \in X\}$. We call it a circuit-graph of $\mathcal{K}$.

Proposition 5.19 For a loop-free closure system $(\mathcal{K}, E)$ and a nontrivial partition $\{E_1, E_2\}$ of $E$, the following are equivalent.

(1) $\mathcal{K}$ is the direct sum of $\mathcal{K}|E_1$ and $\mathcal{K}|E_2$.

(2) $\{E_1, E_2\}$ induces a direct sum decomposition of the circuit graph of $\mathcal{K}$.

(Proof)

(1) $\Rightarrow$ (2) Suppose contrarily that $\{E_1, E_2\}$ does not induce a direct-sum decomposition of the circuit graph. Then there exists an edge connecting a vertex in $E_1$ and a vertex in $E_2$. Without loss of generality, we can assume that there exists a rooted circuit $(X, e)$ of $\mathcal{K}$ satisfying $X \cap E_1 \neq \emptyset$, $e \in E_2$. From Proposition 5.18, $\sigma(X) = \sigma(X \cap E_1) \cup \sigma(X \cap E_2)$, $\sigma(X \cap E_1) \subseteq E_1$, and $\sigma(X \cap E_2) \subseteq E_2$. Since $e \notin \sigma(X \cap E_1)$, $e \in \sigma(X \cap E_2)$ must hold. From the definition of minimality of circuits, $X \cap E_1 = \emptyset$ follows. This contradicts the existence of an edge connecting $E_1$ and $E_2$.

(2) $\Rightarrow$ (1) We shall show that a direct sum decomposition of a circuit graph onto the vertex sets $\{E_1, E_2\}$ will cause a direct sum decomposition of the closure system. From Proposition 5.18, it is sufficient to show that for each $A \subseteq E$,

$$\sigma(A_1) \subseteq E_1, \quad \sigma(A_2) \subseteq E_2, \quad \sigma(A_1 \cup A_2) = \sigma(A_1) \cup \sigma(A_2),$$

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where $A_1 = A \cap E_1$, $A_2 = A \cap E_2$. Now suppose $e \in \sigma(A_1) \setminus A_1$. From Proposition 5.16, there exists a rooted circuit $(X, e)$ such that $X \subseteq E_1$. Since the circuit graph is decomposed onto $\{E_1, E_2\}$, $e \in E_1$ must hold. Hence $\sigma(A_1) \subseteq E_1$ as $e$ is chosen arbitrarily. We can similarly show that $\sigma(A_2) \subseteq E_2$.

Lastly we shall prove $\sigma(A_1 \cup A_2) = \sigma(A_1) \cup \sigma(A_2)$, and complete the proof. Since $\sigma(A_1), \sigma(A_2) \subseteq \sigma(A_1 \cup A_2)$, it is obvious that $\sigma(A_1) \cup \sigma(A_2) \subseteq \sigma(A_1 \cup A_2)$. Let us show the reverse inclusion. Suppose there is an element $e \in E$ such that $e \in \sigma(A_1 \cup A_2) \setminus (A_1 \cup A_2)$. This implies that there is a rooted circuit $(X, e)$ of $K$ with $X \subseteq A_1 \cup A_2$. Suppose $e \in E_1$. By assumption, there is no edge connecting elements in $E_1$ and $E_2$, so that $X \cap A_2 = \emptyset$, that is, $X \subseteq A_1$. Hence we have $e \in \sigma(A_1)$. Similarly $e \in E_2$ leads to $e \in \sigma(A_2)$. As a result, we have established that $\sigma(A_1 \cup A_2) \subseteq \sigma(A_1) \cup \sigma(A_2)$, from which $\sigma(A_1 \cup A_2) = \sigma(A_1) \cup \sigma(A_2)$ follows. This completes the proof. \[ \square \]

**Corollary 5.20** Let $(K, E)$ be a loop-free closure system, and $\{E_1, E_2\}$ to be a non-trivial partition of $E$. Then the following are equivalent.

1. $K = (K|E_1) \oplus (K|E_2)$.
2. For every rooted circuit $(X, e)$, either $X \cup e \subseteq E_1$ or $X \cup e \subseteq E_2$ holds.
Chapter 6

Matroids

6.1 Axioms of matroids

The axioms of matroids are devised by Whitney [157] as an abstraction of linear dependency. For general reference, we refer to Oxley [137] and Welsh [155]. The collection of closed sets of a matroid is a closure system whose closure operator meets the exchange property. Hence a matroid can be considered as a special class of closure systems.

A matroid can be defined in terms of an independence system. Let \( E \) be a finite nonempty set, and \( I \) be a collection of subsets of \( E \).

< Axioms: independent sets >

(I1) \( \emptyset \in I \).

(I2) If \( A' \subseteq A \) and \( A \in I \), then \( A' \in I \).

(I3) If \( X,Y \in I \) and \( |X| > |Y| \), then for some \( x \in X \setminus Y \), \( Y \cup \{x\} \in I \) holds.

If the above axioms are satisfied, \( M = (I,E) \) is a matroid. More precisely, \( I \) is the collection of independent sets of a matroid. As is in an independence system, a subset of \( E \) which is not independent is dependent, and a minimal dependent set is called a circuit. By definition, a set is independent if and only if it does not contain any circuit.

It is obvious from (I3) that the sizes of maximal independent sets in a subset \( A \) are all the same, which is called the rank of \( A \), denoted by \( r(A) \). That is, under assuming (I2), (I3) and (I3') are equivalent.

(I3') For any subset \( A \), the cardinalities of maximal independent sets in \( A \) are all the same.

So we can define a function \( r : A \mapsto r(A) \). The rank function can be represented as

\[
r(A) = \max \{ |X| : X \subseteq A, \ X \in I \},
\]

which we call the rank function of \( M \). Especially, \( r(E) \) is called the rank of a matroid \( M \), and denoted by \( r(M) \).

For two matroids \( M_1 = (I_1, E_1) \) and \( M_2 = (I_2, E_2) \) with \( E_1 \cap E_2 = \emptyset \), \( I_{1,2} = \{ A \cup B : A \in I_1, B \in I_2 \} \) is the collection of a matroid on \( E_1 \cup E_2 \), called the direct-sum of \( M_1 \) and \( M_2 \) and denoted by \( M_1 \oplus M_2 \).
Example 6.1 Examples of matroids.

(1) Let $E$ be a finite nonempty set, and $0 \leq m \leq n = |E|$. Then $U_{m,n} = \{ A \subseteq E : |A| \leq m \}$ is the family of independent sets of a matroid, called a **uniform matroid**.

(2) Let $V$ be a vector space over a field $k$, and $E \subseteq V$ be a nonempty finite set. $I = \{ A \subseteq E : A$ is linearly independent$\}$ is the collection of a matroid. If a matroid can be constructed in this way, it is called **linearly representable over** $k$.

(3) Let $G = (V, E)$ be a graph possibly including parallel edges or self-loops. The collection $\{ A \subseteq E : A$ contains no cycle$\}$ forms the set of independent sets of a matroid on $E$, which we call a graphic matroid of $G$.

In particular, a maximal independent set in the entire set $E$ is called a **base**. Let us denote by $\mathcal{B}$ the collection of all the bases of a matroid on $E$. Then $\mathcal{B}$ satisfies the following:

- **<Axioms: bases>** $(\mathcal{B} \subseteq 2^E)$
- **(B1)** $\emptyset \notin \mathcal{B}$.
- **(B2)** For any $B_1, B_2 \in \mathcal{B}$ and any $x \in B_1 \setminus B_2$, there exists $y \in B_2 \setminus B_1$ such that $(B_1 - x) \cup y \in \mathcal{B}$.

Conversely the collection of subsets of elements of $\mathcal{B}$ constitutes the family of independent sets of the original matroid.

Also a matroid can be defined in terms of rank functions.

- **<Axioms: rank function>** $(r : 2^E \rightarrow \mathbb{Z})$
- **(R1)** $0 \leq r(A) \leq |A|$.
- **(R2)** If $A \subseteq A'$, then $r(A) \leq r(A')$.
- **(R3)** $r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y)$. \hspace{1cm} $(X, Y \subseteq E)$.

For $e \in E$, if $r(\{e\}) = 0$, $e$ is a **self-loop**. For $a_1, \ldots, a_m \in E$ such that each $a_i$ is not a self-loop, if $r(\{a_1, \ldots, a_m\}) = 1$ and $m \geq 2$, then $a_1, \ldots, a_m$ are called **parallel**. If a matroid has no self-loops and no parallel elements, it is called a **simple matroid**.

**Proposition 6.2** The following hold.

(1) The rank function of a matroid satisfies (R1), (R2), and (R3).

(2) Conversely, if an integer-valued function $r : 2^E \rightarrow \mathbb{Z}$ meets (R1), (R2) and (R3), then $I_r = \{ X \subseteq E : r(X) = |X| \}$ is the collection of independent sets of a matroid, and its rank function is equal to $r$.

(Proof of (1)): (R1) and (R2) are obvious. We shall show (R3). Take a maximal independent set in $I_0$ in $X \cap Y$. We can extend $I_0$ to a maximal independent set $I_1$ in $X \cup Y$. Obviously, $|I_1 \cap X| \leq r(X)$ and $|I_1 \cap Y| \leq r(Y)$. Hence, $r(X) + r(Y) \geq |I_1 \cap X| + |I_1 \cap Y| = |(I_1 \cap X) \cap (I_1 \cap Y)| + |(I_1 \cap X) \cup (I_1 \cap Y)| = |I_0| + |I_1| = r(X \cap Y) + r(X \cup Y)$. \hspace{1cm} \square

(Proof of (2)): (I1) is obvious from (R1). We shall show (I2). Suppose $I \in \mathcal{I}$, $I' \subseteq I$. Then (R1) and (R3) signify $|I| = r(I) + r(\emptyset) \leq r(I') + r(I - I') \leq |I'| + |I - I'| = |I|$. Hence all the inequalities hold as equalities, and so $|I'| = r(I')$. 

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We shall show (I3). First assume $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$. Suppose contrarily that for any $e \in I_2 \setminus I_1$, $I_1 \cup e \notin \mathcal{I}$ holds. Since $r(I_1) \leq r(I_1 \cup e) < |I_1 \cup e| = |I_1| + 1$, $r(I_1 \cup e) = r(I_1)$ holds and $r(I_1 \cup I_2) = r(I_1)$ follows from Lemma 6.8. Then we have $|I_1| \geq r(I_1) = r(I_1 \cup I_2) \geq r(I_2) = |I_2|$, a contradiction.

Lastly we should show that the rank function of a matroid $\mathcal{I}_r$ is equal to $r$. We leave the proof to readers.

The collection of the complements of bases of a matroid forms a base system. That is, $\mathcal{B}^* = \{E - B : b \in B\}$ is a collection of a matroid, called the dual matroid of $M$ and denoted by $M^*$. A base and a circuit of $M^*$ are called a cobase and a cocircuit of $M$, respectively.

Although a graph has a dual graph only if it is planar, a matroid always has a dual matroid.

**Lemma 6.3** Let $r$ be the rank function of a matroid. Then if $A \subseteq B$, $r(B) \leq r(A) + |B - A|$.

(Proof) (R1), (R2) The assertion is obvious since it follows from (R1) and (R2) that $r(X \cup e) \leq r(X) + 1$ ($X \subseteq E$, $e \in E$).

The rank function $r^*$ of the dual matroid $M^*$ is determined from the rank function $r$ of $M$.

**Theorem 6.4** Let $\mathcal{B}$ be the family of bases of a matroid $M$ on $E$, and $r$ be the rank function of $M$. Then $\mathcal{B}^* = \{E \setminus B : B \in \mathcal{B}\}$ forms a base family of a matroid. Its rank function $r^*$ is determined from $r$ by

$$r^*(A) = |A| - (r(E) - r(E \setminus A)) = |A| + r(E \setminus A) - r(E) \quad (A \subseteq E).$$

(Proof) We shall firstly show that $r^*$ satisfies (R1), (R2), and (R3).

(R1) It follows from Lemma 6.3 that $r(E) \leq r(E \setminus A) + |A|$. Hence $r^*(A) = r(E \setminus A) + |A| - r(E) \geq 0$. Since $r(E \setminus A) \leq r(E)$, it is obvious that $r^*(A) = r(E \setminus A) + |A| - r(E) \leq |A|$.

(R2) We suppose $A \subseteq A'$, and show $r^*(A) \leq r^*(A')$. From Lemma 6.3, $r(E \setminus A) \leq r(E \setminus A') + |A' - A|$ follows. Hence $r^*(A) = |A| + r(E \setminus A) - r(E) \leq |A'| + r(E \setminus A') - r(E) = r^*(A')$.

(R3) The submodularity of $r^*$ is immediate from the definition.

Since $r(A) \leq r(E)$, $r^*(E \setminus A) = |E| - |A| + r(A) - r(E) \leq |E| - r(E)$. Hence $r^*(E \setminus A) = |E| - r(E)$ is equivalent to $r(A) = r(E)$, and so is equivalent to $A \in \mathcal{B}$. Thus $r^*$ is the rank function of a matroid on $E$, and the collection of bases of this matroid is equal to $\mathcal{B}^* = \{E \setminus A : A \in \mathcal{B}\}$.

**Proposition 6.5** Let $A$ be an independent set of $M$ and $A'$ be an independent set of $M^*$, and suppose $A \cap A' = \emptyset$. Then there exists a base $B \in \mathcal{B}$ of $M$ such that $A \subseteq B$ and $A' \subseteq B' = E \setminus B \in \mathcal{B}^*$.

(Proof) Since $r(E \setminus A') = |E| - |A'| + r^*(A') - r^*(E) = |E| - r^*(E) = r(E)$, the assertion holds.

**Corollary 6.6** For an arbitrary circuit $C$ and a cocircuit $D$ of a matroid, $|C \cap D| \neq 1$ must hold.

(Proof) Suppose there exists a circuit $C$ and a cocircuit $D$ with $|C \cup D| = 1$. Suppose $C \cap D = \{e\}$, and let $A = C - e$ and $A' = D - e$. By Proposition 6.5, there exists a base $B$ such that $A \subseteq B$ and $A' \subseteq E \setminus B = B^*$. Since $\{B, B^*\}$ is a partition of $E$, $e$ belongs to either $B$ or $B^*$. If $e$ is in $B$, then $B$ contains a circuit $C$, a contradiction.

A matroid can be defined also in terms of circuits as follows.
Axioms: circuits

\((\emptyset \neq C \subseteq 2^E)\)

(C1) There exist no \(C_1, C_2 \in \mathcal{C}\) such that \(C_1 \subseteq C_2\).

(C2) For any \(C_1, C_2 \in \mathcal{C}\) and any element \(x \in C_1 \cap C_2\), there exists \(C_3 \in \mathcal{C}\) such that \(C_3 \subseteq (C_1 \cap C_2) - x\).

Given a set system \((\mathcal{C}, E)\) satisfying (C1) and (C2), the collection of subsets of \(E\) including no element of \(\mathcal{C}\) becomes the set of independent sets of a matroid, and vice versa.

We shall next define the closure function of a matroid.

**Lemma 6.7** If \(A \subseteq E, \ x, y \in E, \ x \neq y\) and \(r(A \cup x) = r(A \cup y) = r(A)\), then \(r(A \cup \{x, y\}) = r(A)\).

**(Proof)** Since \(2r(A) = r(A \cup x) + r(A \cup y) \geq r(A \cup \{x, y\}) + r(A) \geq 2r(A)\), every inequality turns out be an equality. Hence the lemma follows. □

Applying Lemma 6.7 repeatedly gives the next corollary.

**Corollary 6.8** In a matroid, suppose \(X, Y \subseteq E\). If any \(e \in Y \setminus X\) satisfies \(r(X \cup e) = r(X)\), then \(r(X \cup Y) = r(X)\).

For \(A \subseteq E\), we shall define

\[\tau(A) = A \cup \{e \in E \setminus A : r(A \cup e) = r(A)\}.\]  \hspace{1cm} (6.3)

It is easy to check that \(\tau\) is a closure function. Furthermore, it follows from Corollary 6.8 that \(r(\tau(A)) = r(A)\) for \(A \subseteq E\). If \(\tau(A) = A\), \(A\) is called a flat or a closed set. We note here that the collection of closed sets of a matroid is a closure system. A proper maximal flat is called a hyperplane. Hence the dimension of the hyperplanes are all equal to \(r(M) - 1\).

A closure function \(\tau: 2^E \to 2^E\) is the closure function of a matroid on \(E\) if and only if

\(<\text{Axiom: closure function}>\)

(Q) If \(x, y \notin \tau(A)\) and \(y \in \tau(A \cup x)\), then \(x \in \tau(A \cup y)\).

This is called the Steinitz-McLane exchange property. For any circuit \(C\) and an element \(e \in C\), \((C \setminus e, e)\) is a rooted circuit with respect to this closure function of a matroid, and vice versa.

Recall that a geometric lattice is an atomistic upper semimodular lattice. The collection of flats of a matroid forms a geometric lattice. Conversely, an arbitrary geometric lattice is isomorphic to the collection of flats of a simple matroid whose underlying set is the set of atoms of the lattice.

### 6.2 Deletions, contractions and minors of matroids

We shall investigate the deletion and the contraction defined in Section 6.2 of the independent sets of a matroid.

Let \(M = (\mathcal{I}, r, E)\) be a matroid, and \(T\) a subset of \(E\).

- \(\mathcal{I}(M|T) = \{X : X \subseteq T, X \in \mathcal{I}\}\) is a set of independent sets of a matroid on \(T\). We denote it \(M|T\), and call it the restriction of \(M\) to \(T\). We also describe it by \(M \setminus (E - T)\), and call it the deletion of \(E - T\).

In other words, a restriction is a matroid whose rank function is \(r^T(X) = r(X)\) (\(X \subseteq T\)).
The following three conditions are equivalent, and define a matroid on \( E - T \) called a contraction of \( T \), denoted by \( M/T \).

1. \( M/T = (M^*|(E \setminus T))^* \).
2. A matroid on \( T \) whose rank function is \( r_T(X) = r(X \cup (E \setminus T)) - r(E \setminus T) \) (\( X \subseteq T \)).
3. The collection of independent sets of \( M/T \) is \( \mathcal{I}(M/T) = \{ X \subseteq T : X \cup B' \in \mathcal{I} \text{ for some } B' \in \mathcal{B}(M|(E \setminus T)) \} \).

For every \( A, B \subseteq E \) with \( A \cap B = \emptyset \), \((M \setminus A)/B = (M/B) \setminus A \) is called a minor of \( M \).

For \( A \cap B = \emptyset \), \(((M \setminus A)/B)^* = (M^*/A) \setminus B \) holds.

### 6.3 2-sum of matroids

Let \( M_1, M_2 \) be matroids on \( E_1 \) and \( E_2 \), respectively. Suppose that \( E_1 \cap E_2 = \{ z \} \) with \( |E_1|, |E_2| \geq 3 \), and that \( z \) is neither a loop nor a coloop of \( M_1, M_2 \). \( C(M_1) \) and \( C(M_2) \) are the collection of the circuits of \( M_1 \) and \( M_2 \), respectively.

We shall define a matroid \( M_1 \Delta M_2 \) on \( E_1 \Delta E_2 \), called a 2-sum, in terms of their circuits as follows.

\[
C(M_1 \Delta M_2) = C_1^* \cup C_2^* \cup C_{1,2},
\]

where

\[
C_1^* = \{ X : X \in C(M_1), z \notin X \},
\]

\[
C_2^* = \{ Y : Y \in C(M_2), z \notin Y \},
\]

\[
C_{1,2} = \{ (X \cup Y) - z : X \in C(M_1), Y \in C(M_2), z \in X, z \in Y \}.
\]

**Proposition 6.9 ([137])** \( C(M_1 \Delta M_2) \) is the collection of circuits of a matroid on \( E = E_1 \Delta E_2 \), denoted \( M_1 \Delta M_2 \).

### 6.4 Fundamental classification of matroids

We shall define subclasses of matroids.

- A matroid is a **binary matroid** if it is linearly representable over the finite field \( GF(2) \).
- A matroid is a **regular matroid** if it is representable by a totally unimodular matrix over \( \mathbb{R} \). A matrix is totally unimodular if the determinant of every square submatrix is 0 or ±1.
- In an undirected graph, the collection of sets of edges containing no circuit forms a collection of independent sets of a matroid, which is called a graphic matroid. A cographic matroid is a dual matroid of a graphic matroid. In case of a graphic matroid of a graph, a rooted circuit \((C \setminus e, e)\) is a critical circuit if and only if \( C \) is a chordless cycle.
- A matroid is planar if it is a graphic matroid of a planar graph.
We consider a vector space over $GF(2)$ of dimension 3. The column vectors of the matrix

$$M = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}$$

constitute a matroid with respect to linear independency, which we call the Fano matroid and denote by $F_7$.

$F_7$ is also considered to be a projective plane, called a Fano geometry, consisting of 7 points and 7 lines, illustrated in Fig. 6.1 where each point corresponds to an element and each line corresponds to a hyperplane of the Fano matroid.

**Theorem 6.10 ([137, 155])** For a matroid $M = (E, I)$, the following are equivalent.

1. $M$ is a binary matroid.
2. $M$ contains no minor isomorphic to $U_{2,4}$.
3. If $C$ is a circuit and $D$ is a cocircuit, $|C \cap D|$ is necessarily even.
4. If $C_1$, $C_2$ are distinct circuits, then $C_1 \Delta C_2$ is a disjoint union of circuits.

Hence the dual of a binary matroid is also binary.

**Theorem 6.11 ([137, 155])** For a matroid $M$, the following are equivalent.

1. $M$ is a regular matroid.
2. $M$ is binary, and contains no minor isomorphic to $F_7$ or $F_7^*$.
3. $M$ is linearly representable over every field.

**Theorem 6.12 ([137, 155])** Let $M(K_5)$ and $M(K_{3,3})$ be graphic matroids of the complete graph $K_5$ and the complete bipartite graph $K_{3,3}$, respectively. The following hold.

1. A regular matroid is graphic if and only if it contains a minor isomorphic to neither $M(K_5)^*$ nor $M(K_{3,3})^*$.
2. A regular matroid is cographic if and only if it contains a minor isomorphic to neither $M(K_5)$ nor $M(K_{3,3})$. 

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A regular matroid is planar if and only if it is both graphic and cographic.

These classes of matroids have the inclusion relations illustrated in Fig. 6.2. Every inclusion is proper.

**Proposition 6.13** For a matroid on \( E \), \( H \) is a hyperplane if and only if \( E - H \) is a cocircuit.

**(Proof)** Let \( H \) be a hyperplane of a matroid \( M \) on \( E \). Since \( H \) does not contain a base of \( M \), there is no cobase which includes \( E - H \). Hence \( D = E - H \) is a dependent set of the dual matroid \( M^* \). \( H \) is maximal with respect to these conditions. Hence \( D \) is a minimal dependent set of \( M^* \). That is, \( D \) is a cocircuit of \( M \). The reverse direction is similarly shown. \( \square \)

### 6.5 Characteristic polynomials and \( \beta \)-invariants of matroids

Let \( M \) be a simple matroid on a finite nonempty set \( E \), and \( L(M) \) be the geometric lattice consisting of the flats (closed sets) of \( M \). Replacing \( P \) in (4.8) and (4.9) with \( L(M) \), we have the corresponding characteristic polynomial \( p(L(M); \lambda) \) and the \( \beta \)-invariant \( \beta(L(M)) \). For simplicity, we denote them by \( p(M; \lambda) \) and \( \beta(M) \), respectively. Then

\[
p(M; \lambda) = \sum_{X \in L(M)} \mu_{L(M)}(\emptyset, X)\lambda^{r(E) - r(X)}, \quad (6.4)
\]

\[
\beta(M) = -\frac{d}{d\lambda} \frac{p(M; \lambda)}{d\lambda} \bigg|_{\lambda=1} = \sum_{X \in L(M)} \mu_{L(M)}(\emptyset, X)r(X), \quad (6.5)
\]

where \( r \) is the rank function of \( M \), and \( \mu_{L(M)} \) is the Möbius function of the semimodular lattice \( L(M) \). Particularly, if \( M \) consists of loops only, then \( p(M; \lambda) = 0 \) and \( \beta(M) = 0 \), and if \( M \) consists of \( n \) coloops only, then \( p(M; \lambda) = (\lambda - 1)^n \) and \( \beta(M) = (-1)^n \).

Historically, the notion of characteristic polynomials stems from the chromatic polynomials of graphs, and hence they are directly connected.

The characteristic function of a ranked poset has the origin in the chromatic polynomials of graphs. In case of a graphic matroid of a graph \( G \), the characteristic function of the graphic matroid \( M(G) \) is almost equal to the chromatic polynomial of \( G \).
Lemma 6.14 For a simple graph \( G \) and a \( \lambda \)-coloring \( f : V(G) \to [\lambda] \), \( F(f) = \{ \{ u, v \} \in E(G) : f(u) = f(v) \} \) is a flat of the graphic matroid \( M(G) \). Furthermore, \( f \) is a proper coloring if and only if \( F(f) = \emptyset \).

Theorem 6.15 ([2, 144]) Let \( G \) be a simple graph, and \( c(G) \) be the number of the connected components of \( G \). Let \( \chi(G; \lambda) \) be the chromatic polynomial of \( G \). Let \( M(G) \) be the graphic matroid on the edge set \( E(G) \), and \( r \) be the rank function of \( M(G) \). \( L_G \) denotes the lattice of flats of \( M(G) \), and \( p(M(G); x) \) is the characteristic function of matroid \( M(G) \). Then

\[
\chi(G; \lambda) = \lambda^{c(G)} p(M(G); \lambda).
\]

(Proof) For \( A, B \in L_G \), we define

\[
f(A; \lambda) = |\{ f : V(G) \to [\lambda] : F(f) = A \}|
\]
and

\[
g(B; \lambda) = \sum_{X \in L_G : B \subseteq X} f(X; \lambda).
\]

If a coloring \( f \) is constant on each connected components of the subgraph \( G|B \), it gives a coloring on \( G/B \). This is a one-to-one correspondence. Since \( |V(G)| = |V(G/B)| + r(B) \), the following holds.

\[
|\{ F(f) : B \subseteq F(f) \}| = \text{the number of } \lambda\text{-coloring on } G/B,
\]

\[
= \lambda^{|V(G/B)|} = \lambda^{|V| - r(B)}
\]

Hence

\[
g(B; \lambda) = \lambda^{|V| - r(B)} = \sum_{X \in L_G : B \subseteq X} f(X; \lambda). \tag{6.6}
\]

Applying the Möbius inversion theorem 4.24 to (6.6) leads to the following.

\[
f(A; \lambda) = \sum_{X \in L_G : A \subseteq X} \mu(A, X) g(X; \lambda) = \sum_{X \in L_G : A \subseteq X} \mu(A, X) \lambda^{|V(G)| - r(X)}, \quad (A \in L_G)
\]

Now \( |V(G)| = r(E(G)) + c(G) \) holds. Hence

\[
\chi(G; \lambda) = f(\emptyset; \lambda) = \sum_{X \in L_G} \mu(\emptyset, X) \lambda^{|V(G)| - r(X)} = \lambda^{c(G)} \sum_{X \in L_G} \mu(\emptyset, X) \lambda^{r(E(G)) - r(X)}
\]

\[
= \lambda^{c(G)} p(M(G); \lambda).
\]

\[
\square
\]

Crapo’s \( \beta \)-invariant [46], denoted by \( \tilde{\beta}(M) \), relates to our \( \beta \)-invariant by \( \tilde{\beta}(M) = (-1)^{r(E)} \beta(M) \).

Proposition 6.16

1. \( \tilde{\beta}(M) = 0 \) if and only if \( M \) is disconnected,
2. \( \tilde{\beta}(M) = 1 \) if and only if \( M \) is a series-parallel graph.

The following (I), (II), (III), and (IV) are the basic properties of broken circuit complexes, characteristic polynomials, and \( \beta \)-invariants of matroids.
(I) Broken circuit complex and Brylawski’s decomposition:

Let $M$ be a matroid on $E$, and $\omega$ be an arbitrary linear order on the underlying set $E$. For a circuit $C$ of $M$, the set $C - \min_\omega C$ obtained by deleting the smallest element from $C$ is called a broken circuit of $(M, \omega)$. A set is nbc-independent if it does not contain any broken circuit. The collection of all the nbc-independent sets forms a simplicial complex $BC(M, \omega)$, called a broken circuit complex. A broken circuit complex is a pure $(r - 1)$-dimensional complex, and a subcomplex of $\text{In}(M)$ of all the independent sets. For more details, we refer to Björner [22].

A useful decomposition theorem of broken circuit complexes is shown by Brylawski [37].

**Theorem 6.17 ([37])** Let $x$ be the maximum element with respect to $\omega$. Then

$$BC(M, \omega) = BC(M \setminus x, \omega) \cup (BC(M/x, \omega) * x) \quad (6.7)$$

where $BC(M/x, \omega) * x = \{ X \cup x : X \in BC(M/x, \omega) \}$.

(II) Whitney-Rota’s formula:

The broken circuit complex $BC(M, \omega)$ fixes the coefficients of the characteristic polynomial, which is known as Whitney-Rota’s formula)

**Theorem 6.18 (Rota [144])** For a matroid $M$ on $E$ and any total order $\omega$ on $E$,

$$p(M; \lambda) = \sum_{X \in BC(M, \omega)} (-1)^{|X|} \lambda^{r(E) - r(X)}. \quad (6.8)$$

Hence the right-hand side of (6.8) does not depend on the choice of $\omega$.

The broken circuit complexes are further investigated by Björner and Ziegler [27] in which they have presented the sufficient conditions for a broken circuit complex of a matroid factors completely. The details will be stated in Section 6.6.

(III) Boolean expansions:

$p(M; \lambda)$ and $\beta(M)$ have the Boolean expansions as (See, for example, [163])

$$p(M; \lambda) = \sum_{A \in 2^E} (-1)^{|A|} \lambda^{r(E) - r(A)}, \quad (6.9)$$

$$\beta(M) = \sum_{A \in 2^E} (-1)^{|A|} r(A). \quad (6.10)$$

(IV) The direct-sum factorization and the deletion-contraction rule (See [163]):

1. If $M = M_1 \oplus M_2$, then

$$p(M; \lambda) = p(M_1; \lambda)p(M_2; \lambda). \quad (6.11)$$

2. The characteristic polynomial and the $\beta$-invariant of a matroid $M$ follow the deletion-contraction rules below.

   (a) If $e \in E$ is not a coloop of $M$, then

   $$p(M; \lambda) = -p(M/e; \lambda) + p(M \setminus e; \lambda), \quad (6.12)$$

   $$\beta(M) = -\beta(M/e) + \beta(M \setminus e). \quad (6.13)$$
(b) If $e$ is a coloop of $M$ and $E \setminus e \neq \emptyset$, then

$$p(M; \lambda) = (\lambda - 1)p(M \setminus e; \lambda),$$  \hspace{1cm} (6.14)

$$\beta(M) = 0.$$  \hspace{1cm} (6.15)

### 6.6 Supersolvable matroids

When the lattice of flats of a matroid $M$ is supersolvable, $M$ is said to be *supersolvable*.

For the lattice of a supersolvable matroid, Theorem 4.27 and the factorization of (4.10) are restated as follows.

**Corollary 6.19** Let $L$ be the geometric lattice of flats of a supersolvable simple matroid $M$ of rank $r$ with an $M$-chain $\emptyset = M_0 < M_1 < \cdots < M_r = E$. Let $e_i = |M_i - M_{i-1}|$ $(1 \leq i \leq r)$. Then

$$p(L(M); \lambda) = (\lambda - e_1)(\lambda - e_2)\cdots(\lambda - e_r)$$

A broken circuit complex $BC(M, \omega)$ is said to *factor completely* if the characteristic function of the matroid $M$ is decomposed into the products of linear terms with integer roots.

**Theorem 6.20 (Björner and Ziegler [27])** For a loop-free matroid $M$, the following are equivalent.

1. $M$ is supersolvable.
2. For some linear order $\omega$ on $E$, the broken-circuit complex $BC(\omega)$ factors completely.
3. For some linear order $\omega$ on $E$, the 1–skeleton $BC(M, \omega)^[1]$ is a complete $r$-partite graph.
4. For some linear order $\omega$ on $E$, the minimal broken circuits (with respect to inclusion) all have size 2.
5. There exists an ordered partition $(X_1, X_2, \ldots, X_r)$ of $E$ such that if $x, y \in X$, $x \neq y$, then there exists $z \in X_j$ with $j < i$ such that $\{x, y, z\}$ is a circuit.

For a chordal graph $G$ and any total order $\omega$ on $E(G)$, the minimal broken circuit are all of size two. Hence Theorem 6.20 (4) can be applied to it.

**Corollary 6.21 (Stanley [149])** Let $M(G)$ be a graphic matroid of a graph $G$. If $G$ is a chordal graph, then $M(G)$ is a supersolvable matroid.

Hence as is shown in Corollary 6.19, the characteristic function of a graphic matroid of a chordal graph factors to the product of linear terms with non-negative integer roots. At the same time, it is proved in Theorem 6.15 that the chromatic polynomial $\chi(G; \lambda)$ of a graph $G$ is equal to the characteristic polynomial of the graphic matroid $M(G)$ multiplied by $\lambda^{c(G)}$.

Hence

**Corollary 6.22** The chromatic polynomial $\chi(G; \lambda)$ of a chordal graph $G$ factors to the product of linear terms with non-negative integer roots.
Corollary 6.22 was already explained in p.11 through simplicial shelling. Though Corollary 6.22 can be deduced directly from the perfect simplicial shelling, we have taken these processes as we would like to show that the factorization of the chromatic polynomial of a chordal graph is a special example of a wider framework of Theorem 6.20.

We will consider the broken circuit complex and the characteristic polynomial of a convex geometry with the broken circuits (stems) all of size two in Section 11.4.

6.7 Tutte polynomials and rank generating functions of matroids

A Tutte-Grothendieck invariant defined by Brylawski [36] is a function $f$ from the class of matroids to a commutative ring meeting the following conditions.

1. If a matroid $M_1$ is isomorphic to a matroid $M_2$, then $f(M_1) = f(M_2)$.
2. $f(M) = f(M \setminus e) + f(M/e)$ for any element $e$ of $M$ which is neither a loop nor a coloop.
   (Deletion-contraction rule)
3. For a direct-sum $M = M_1 \oplus M_2$ of matroids, $f(M) = f(M_1) f(M_2)$.

The rank generating function $R(M; x, y)$ and the Tutte polynomial $T(M; x, y)$ of a matroid $M = (I, r, E)$ are defined as below.

$$R(M; x, y) = \sum_{A \subseteq E} x^{r(E)} y^{r(A)} - r(A) = \sum_{A \subseteq E} x^{r(E) - r(A) y^{r(A)} - r(A)}, \quad (6.16)$$

$$T(M; x, y) = R(M; x - 1, y - 1). \quad (6.17)$$

It is easy to calculate the following.

1. $T(M; 2, 2) = R(M; 1, 1) = 2^{|E|}$,
2. $T(M; 1, 1) = R(M; 0, 0) = $ the number of bases of $M$,
3. $T(M; 2, 1) = R(M; 1, 0) = $ the number of independent sets,
4. $T(M; 1, 2) = R(M; 0, 1) = $ the number of spanning sets,
5. $T(M; 0, 0) = R(M; -1, -1) = 0$.

**Proposition 6.23 (Deletion-contraction rule [155])** Let $M = (I, r, E)$ be a matroid. Then for an element $e \in E$, the following hold.

$$R(M; x, y) = \begin{cases} 
(1 + y) R(M \setminus e; x, y) & \text{if } e \text{ is a loop,} \\
(1 + x) R(M/e; x, y) & \text{if } e \text{ is a coloop,} \\
R(M \setminus e; x, y) + R(M/e; x, y) & \text{otherwise.}
\end{cases}$$

$$T(M; x, y) = \begin{cases} 
y T(M \setminus e; x, y) & \text{if } e \text{ is a loop,} \\
x T(M/e; x, y) & \text{if } e \text{ is a coloop,} \\
T(M \setminus e; x, y) + T(M/e; x, y) & \text{otherwise.}
\end{cases}$$
Proposition 6.24 ([155]) For a direct sum $M_1 \oplus M_2$ of matroids, it holds that

$$R(M_1 \oplus M_2; x, y) = R(M_1; x, y)R(M_2; x, y),$$
$$T(M_1 \oplus M_2; x, y) = T(M_1; x, y)T(M_2; x, y).$$

Theorem 6.25 The Tutte polynomial as well as the rank generating function is a Tutte-Grothendieck invariant.

The Tutte polynomial as well as the rank generating function is universal in the sense that every Tutte-Grothendieck invariant is an evaluation of the Tutte-polynomial.

Theorem 6.26 (Brylawski [36], [155, 156]) Let $f$ be an arbitrary Tutte-Grothendieck invariant on matroids, and $a = f(a\text{ coloop}), b = f(a\text{ loop})$. Then $f(M) = T(M; a, b)$.

For the characteristic polynomial $p(M; x)$ of a matroid $M$, $(-1)^{r(M)}p(M; x)$ is seen to be a Tutte-Grothendieck invariant from (6.11) and (6.12). Hence $(-1)^{r(M)}p(M; x)$ is an evaluation of the Tutte polynomial. In another way, considering (6.9), (6.10) and (6.16), the direct calculation shows the following formulae.

Proposition 6.27 For the characteristic function $p(M; \lambda)$ and the beta-invariant $\beta(M)$ of a matroid, they are evaluated from the rank generating function $R(M; x, y)$ and the Tutte polynomial $T(M; x, y)$ as follows.

$$p(M; \lambda) = (-1)^{r(M)}R(M; -\lambda, -1) = (-1)^{r(M)}T(M; 1 - \lambda, 0),$$
$$\beta(M) = (-1)^{r(M)}\frac{\partial R(M; x, y)}{\partial x}(-1, -1) = (-1)^{r(M)}\frac{\partial T(M; x, y)}{\partial x}(0, 0).$$

6.8 Oriented matroids

An oriented matroid is an oriented version of a matroid. The set of circuits of an undirected graph determines a matroid on the edge set. Analogously, the set of directed circuits of a directed graph defines a directed version of matroid, called an oriented matroid. For oriented matroids, we refer to [24].

In the directed graph of Fig.6.3, $C = \{1, 2, 3, 4\}$ is a cycle in the sense of an undirected graph. This cycle gives a pair of signed sequence tracing clockwise and anti-clockwise, $Y_1 = (-, +, +, +, 0)$ and $Y_2 = (+, -, -, -, 0)$, where + and − mean a forward edge and a backward edge in each directed cycle, respectively.

A signed set on an underlying set $E$ is an ordered pair $X = (X^+, X^-)$ such that $X^+, X^- \subseteq E$ and $X^+ \cap X^- = \emptyset$. A sequence $Y_1$ of symbols $\{\pm, 0\}$ of our example corresponds to a signed set $X_1 = (X_1^+, X_1^-)$ where $X_1^+ = \{2, 3, 4\}$ and $X_1^- = \{1\}$.

There exist several ways to define an oriented matroid. An oriented matroid $\mathcal{M} = (C, E)$ can be defined in terms of a family $C$ of signed sets satisfying below.

< Axioms of oriented matroid: circuits >
(1) $\emptyset = (\emptyset, \emptyset) \notin C$.

(2) $X = (X^+, X^-) \in C \implies -X = (X^-, X^+) \in C$.

(3) No proper subset of a circuit is a circuit.

(4) If $X_1, X_2 \in C$, $X_1 \neq -X_2$, and $e \in X_1^+ \cap X_2^-$, then there exists $Y = (Y^+, Y^-) \in C$ such that $Y^+ \subseteq (X_1^+ \cup X_2^+) \setminus e$ and $Y^- \subseteq (X_1^- \cup X_2^-) \setminus e$.

An oriented matroid is acyclic if it does not contain a signed set $(X^+, \emptyset)$. It is called totally cyclic provided that for any element $e$, there exists $(X^+, X^-)$ such that $e \in X^+$, $X^- = \emptyset$.

For an oriented matroid $\mathcal{M} = (\mathcal{C}, E)$, the collection of sets $X^+ \cup X^-$ of the signed sets is a family of circuits of a matroid, which we shall call the underlying matroid of $\mathcal{M} = (\mathcal{C}, E)$. Conversely, a matroid $M$ is said to be orientable if there exists an oriented matroid whose underlying matroid is $M$.

**Proposition 6.28 (Bland and Las Vergnas [30])** A binary matroid is orientable if and only if it is a regular matroid.

**Example 6.29** Examples of oriented matroids.

1. **$\mathbb{R}$–linear representable oriented matroid**: Suppose that a set of non-zero vectors $v_1, \ldots, v_n \in \mathbb{R}^n$ is given. For each minimal linear dependent relation
   \[ \sum_i \lambda_i v_i = 0, \]
   we set $X^+ = \{i : \lambda_i > 0\}$, $X^- = \{i : \lambda_i < 0\}$ and define a signed set $(X^+, X^-)$. These signed sets constitute an oriented matroid, called an $\mathbb{R}$–linear representable oriented matroid.

2. **Affine realizable oriented matroids**: Let $E \subseteq \mathbb{R}^d$ be an affine point configuration. Suppose $\{v_1, \ldots, v_k\} \subseteq E$ to be a minimal affine dependent set, and
   \[ \sum_{i=1}^k \lambda_i v_i = 0, \quad \sum_{i=1}^k \lambda_i = 0. \]
   This determines a signed set $(X^+, X^-)$ where $X^+ = \{v_i : \lambda_i > 0\}$ and $X^- = \{v_i : \lambda_i < 0\}$. The collection of signed sets arising from these minimal affine dependent equations forms an oriented matroid, called an affine realizable oriented matroid. An affine realizable oriented matroid is necessarily acyclic, and the class of affine realizable oriented matroids equals to that of acyclic $\mathbb{R}$–linear representable oriented matroids.
We shall mention in Section 8 that an acyclic oriented matroid $\mathcal{M}$ defines a convex geometry. Lastly we remark the relation of classes of acyclic oriented matroids.

acyclic directed graph $\subseteq$ regular acyclic oriented matroid

$\subseteq$ linear representable acyclic oriented matroid = affine realizable oriented matroid.
Chapter 7

Antimatroids and Convex Geometries

7.1 Shelling process and antimatroids

We shall present first the axiom sets of antimatroids together with their variations. We start with defining shelling process to explain what an antimatroid is like.

We shall prepare some terminology of languages. For a nonempty finite set $E$, a sequence $e_1 \cdots e_k$ of elements is called a word. $k$ is the length of the word. $\epsilon$ denotes the word of null length. $E^*$ denotes the collection of all the words of $E$. For a word $\alpha = a_1 \cdots a_k \in E^*$, the set of elements in $\alpha$ is denoted by $[\alpha] = \{a_1, \ldots, a_k\}$. For a word $\alpha = a_1 \cdots a_m$, $a_1 \cdots a_j$ for $0 \leq j \leq m$ is called a prefix of $\alpha$. A collection of words is called a language. A word is simple if it contains no repetition of an element. $E^*_S$ is the set of all the simple words of $E$. A language is simple if it consists of simple words only. A language is hereditary if any prefix of a word in the language also belongs to it.

An elimination process is a procedure in which the elements of a set is removed one at a time according to some rule. Finally, every elimination process gives rise to a sequence (word) of elements in its deleted order. Hence the collection of all the prefixes of the words derived in this way is a hereditary simple language.

A little bit more precisely, suppose that we are given a nonempty finite set $E$ and a map $f : 2^E \rightarrow 2^E$ with $f(A) \subseteq A$ for every $A \subseteq E$. We consider that the elements in $f(A)$ are the removable elements in $A$. ($f$ is called a choice function. The theory of choice function will be describe later in Chapter 14.) An elimination process is the following procedure:

\[
\begin{align*}
X &:= E; \quad [\text{set}] \\
\alpha &:= \epsilon; \quad [\text{string}]
\end{align*}
\]

while $f(X) \neq \emptyset$ do begin

Choose an arbitrary element $e$ in $f(X)$;

\[
\begin{align*}
\alpha &:= \alpha e; \quad X := X - e;
\end{align*}
\]

end.

A shelling process is an elimination process such that once an element becomes removable at some step of the process, it remains removable at any later step. A shelling process can be characterized in terms of choice functions as follows.

\[
\begin{align*}
\text{(H)} & \quad \text{If } a \in B \subseteq A, \text{ and } a \in f(A), \text{ then } a \in f(B). \quad (7.1)
\end{align*}
\]
The collection of all the prefixes of the sequences produced by the possible shelling processes is an antimatroid language. (See [8, 21, 110].) By definition, an antimatroid language is a simple hereditary language. We do not assume that all the elements of the underlying set are removed at the end of a shelling process. That is, we may admit the existence of loops or dummy elements which are the elements left after finishing a shelling process.

We can also express the shelling process in terms of hereditary language $L$.

If $\alpha \in L$, $\alpha \beta \in L$, and $x \not\in [\beta]$, then $\alpha \beta x \in L$. \hfill (7.2)

Now we can axiomatize the notion of antimatroid languages.

< Axioms of antimatroid languages > \hfill ($L \subseteq E^*_S$)

(L0) $\epsilon \in L$.

(L1) Every prefix of a word in $L$ is also in $L$.

(L2) If $\alpha, \alpha x, \alpha y \in L$ and $x, y \in E - [\alpha]$, then $\alpha yx \in L$.

Since we can deduce (7.2) using (L2) repeatedly, (7.2) is equivalent to (L2). A simple language $L$ on a non-empty finite set $E$ is an antimatroid language if it satisfies (L0), (L1), and (L2). An antimatroid is equivalently defined as a collection $\{[\alpha] : \alpha \in L\}$ of subsets for an antimatroid language $L$. Namely, a family $F \subseteq 2^E$ is an antimatroid if it satisfies the following.

< Axioms of antimatroids >

(A0) $\emptyset \in F$,

(AC) If $X \in F$ and $X \neq \emptyset$, then there exists $e \in X$ such that $X - e \in F$.

(UC0) If $X, X \cup a, X \cup b \in F$, then $X \cup \{a, b\} \in F$.

An element of an antimatroid is called a feasible set. Under assuming (A0) and (AC), (UC0) is equivalent to

(UC) $X, Y \in F \implies X \cup Y \in F$,

Hence we can rewrite the axiom sets of antimatroids.

< Axioms of antimatroids >

(A0) $\emptyset \in F$,

(UC) $X, Y \in F \implies X \cup Y \in F$,

(AC) If $X \in F$ and $X \neq \emptyset$, then there exists $e \in X$ such that $X - e \in F$. \hfill (accessibility)

By (AC), for each $Y \in F$, there exists an elementary chain from $\emptyset$ to $Y$;

$$\emptyset = X_0 \subsetneq X_1 \cdots \subsetneq X_n = Y \quad (X_i \in F \text{ such that } |X_i - X_{i-1}| = 1 \text{ for } 1 \leq i \leq n).$$ \hfill (7.3)

Under (UC), (AC) is equivalent to a strong accessibility (AC').

(AC') For any $X, Y \in F$ with $X \subseteq Y$, there exists an elementary chain in $F$ from $X$ to $Y$. \hfill (strong accessibility)
(Proof) Take an elementary chain \( \{ Y_j \}_{j=0}^{r} \) from \( \emptyset \) to \( Y \). By (UC), \( X \cup Y \in \mathcal{F} \) holds. Hence \( \{ X \cup Y_j \}_{j=0}^{r} \) includes an elementary chain from \( X \) to \( Y \). \( \square \)

We shall check the equivalence of antimatroid languages and antimatroids.

**Proposition 7.1** Let \( \mathcal{L} \subseteq E^*_s \) be an antimatroid language. Then the collection \( \mathcal{A}_\mathcal{L} = \{ [\alpha] : \alpha \in \mathcal{L} \} \) is an antimatroid. \( \mathcal{A}_\mathcal{L} \) and \( \mathcal{L} \) are equivalent in the sense that there is a one-to-one correspondence between elementary chains in \( \mathcal{A}_\mathcal{L} \) and words in \( \mathcal{L} \). Namely, let \( \emptyset = A_0 \subsetneq A_1 \subsetneq A_2 \cdots \subsetneq A_k = A \in \mathcal{A} \) with \( A_i \in \mathcal{A} \) and \( a_i = A_i - A_{i-1} \) for \( i = 1, \ldots, k \). Then \( a_1a_2 \cdots a_k \in \mathcal{L} \) and vice versa.

(Proof) For any antimatroid language, it is obvious that \( \mathcal{A}_\mathcal{L} \) satisfies (A0), (AC), and (UC0).

Conversely, take a maximal chain in \( \mathcal{A}_\mathcal{L} \): \( \emptyset = A_0 \subsetneq A_1 \subsetneq A_2 \cdots \subsetneq A_k = A \in \mathcal{A} \). By (AC'), it must be an elementary chain. Hence we can define \( \{ a_i \} = A_i - A_{i-1} \) for \( i = 1, \ldots, k \). We shall show \( \alpha = a_1a_2 \cdots a_k \in \mathcal{L} \). We use induction on \( k \). If \( k = 1 \), the assertion is trivial. Suppose \( k > 1 \). By definition, there exists a word \( \alpha' = a'_1 \cdots a'_k \in \mathcal{L} \) such that \( A_k = [\alpha'] \). By induction hypothesis, \( a'_j = a_j \) for \( j = 1, \ldots, k - 1 \) whereas \( [\alpha'] = [\alpha] = A_k \). Hence \( a'_k = a_k \). This completes the proof. \( \square \)

Using (AC') and (UC), we can deduce a strong augmentation property below.

\[ (\text{AG}) \quad X, Y \in \mathcal{F}, \ X \not\subseteq Y \Rightarrow \exists x \in X \setminus Y : Y \cup x \in \mathcal{F}. \] (strong augmentation property)

(Proof) Suppose \( X, Y \in \mathcal{F}, \ X \not\subseteq Y \), and let \( Z \in \mathcal{F} \) be a maximal feasible set in \( X \cap Y \). From (AC'), there exists an elementary chains of feasible sets from \( Z \) to \( X \). In particular, there is an element \( x \in X \setminus Y \) such that \( Z \cup x \in \mathcal{F} \). It follows from (UC) that \( Y \cup (Z \cup x) = Y \cup x \in \mathcal{F} \). Hence (AG) is shown. \( \square \)

Conversely, (AG) imply (UC) and (AC') both, and (AC) follows from (A0) and (AC'). Hence a pair of (A0) and (AG) forms an equivalent axiom set.

**< Axioms of antimatroids >** \( \mathcal{L} \subseteq E^*_s \)

(0) \( \emptyset \in \mathcal{F} \).

(AG) If \( X, Y \in \mathcal{F}, \ X \not\subseteq Y \), then there exists an element \( x \in X \setminus Y \) such that \( Y \cup x \in \mathcal{F} \).

Thus the above set of axioms gives that of antimatroid languages.

**< Axioms of antimatroid languages >** \( \mathcal{L} \subseteq E^*_s \)

(L0) \( \epsilon \in \mathcal{L} \).

(AG) If \( \alpha, \beta \in \mathcal{L}, \ [\alpha] \not\subseteq [\beta] \), then there exists an element \( x \in [\alpha] \setminus [\beta] \) such that \( \beta x \in \mathcal{L} \).

Greedoids has been introduced by Korte and Lovász [108] as a broad framework for greedy algorithms. Greedoids include as a special case both the collection of independent sets of a matroid and an antimatroid. In [79], 42 examples of greedoids are listed.

**<Axioms of Greedoid >**

(A0) \( \emptyset \in \mathcal{F} \).

(AG) \( X, Y \in \mathcal{F}, \ |X| > |Y| \Rightarrow \exists x \in X \setminus Y : Y \cup x \in \mathcal{F} \).
7.2 Convex geometries

Let \( E \) be a nonempty finite set. The collection of the complement sets of feasible sets of an antimatroid is a *convex geometry*, and vice versa. An antimatroid and a convex geometry are said to be *dual* to each other when they are in one-to-one correspondence by taking the complement. Hence the notions of antimatroid and convex geometry are cryptomorphic.

Precisely, \( \mathcal{K} \subseteq 2^E \) is a *convex geometry* if it satisfies

< Axioms of convex geometries: closure system >

1. \( E \in \mathcal{K} \),
2. \( X, Y \in \mathcal{K} \implies X \cap Y \in \mathcal{K} \),
3. \( X \in \mathcal{K}, X \neq E \implies \exists e \in E \setminus X : X \cup e \in \mathcal{K} \). (accessibility)

That is, a convex geometry is a closure system which meets the accessibility. An element of a convex geometry is called a *closed set* or a *convex set*.

An antimatroid and the set of independent sets of a matroid are special cases of greedoids [108, 109, 112]. A convex geometry and the geometric lattice of closed sets of a matroid are special cases of closure systems. We emphasize that we follow the second line of common generalization as closure systems hereafter.

For the axioms of antimatroids and convex geometries, please also refer to Björner and Ziegler [28], Edelman and Jamison [63], and Korte, Lovász and Schrader [112].

\[ \text{closure system} \quad \text{antisymmetry of } \cup \]

\[ \text{flats of a matroid} \quad \text{convex geometry} \quad \text{anti-matroid geometry} \quad \text{independent sets of a matroid} \quad \text{dual (complement)} \]

\[ \text{A convex geometry can be also defined in terms of the closure operator of a closure system.} \]

< Axioms of convex geometries: closure operator >

1. \( (\mathcal{K}, \sigma, E) \) is a closure space,
2. if \( x \neq y, x, y \notin \sigma(A) \) and \( y \in \sigma(A \cup x) \), then \( x \notin \sigma(A \cup y) \). (\( x, y \in E, A \subseteq E \) anti-exchange property)

The equivalence of these two axioms will be stated in Theorem 7.7.

The anti-exchange property can be understood as an abstraction of the convexity in affine spaces and vector spaces. You can just take an ordinary convex hull in a Euclidean space as a closure operator. Suppose that \( E \) is a finite set of points in a Euclidian space \( \mathbb{R}^n \). Then \( \sigma(A) = \text{conv.lull}(A) \cap E \) for \( A \subseteq E \) is a closure operator on \( 2^E \) naturally satisfying the anti-exchange property. It is illustrated in Fig. 7.2. You see \( x, y \notin \sigma(\{a, c, e\}) = \{a, b, c, d, e\} \) and \( y \in \sigma(\{a, c, e, x\}) \), but \( x \notin \sigma(\{a, c, e, y\}) \).
Theorem 7.2 (Edelman [59]) Let $K_i \subseteq 2^E$ be a convex geometry with the closure operator $\sigma_i$ for $i = 1, \ldots, m$.

Then $K = K_1 \lor K_2 \lor \cdots \lor K_m = \{X_1 \cap \cdots \cap X_m : X_i \in K_i$ for $i = 1, \ldots, m\}$ is a convex geometry on $E$ whose closure operator is $\sigma(A) = \cap_{i=1}^m \sigma_i(A)$.

Example 7.3 Fig. 7.3 shows four points on a plane. Now the underlying set is $E = \{1, 2, 3, 4\}$. The closure operator is $\sigma(A) = \text{conv.hull}(A)$ for $A \subseteq E$ where $\text{conv.hull}(A)$ is the convex hull of $A$ in the ordinary sense, and the extreme-point operator $\text{ex}(A)$ = the set of vertices of $\text{conv.hull}(A)$ . The shelling is a deletion of a vertex of the convex-hull of the remaining points. At first you can delete 1, 2 or 3, but not 4. 4 is undeletable until 1 or 3 is deleted. Fig. 7.4 is the resultant antimatroid. As is easily checked, the set of shelling chains of Fig. 7.3 coincides with the collection of maximal chains of Fig. 7.4.

Since an antimatroid $\mathcal{F}$ is closed under union, for each subset $A$, there is uniquely the maximum feasible set in it, which we call the base of $A$. (Note that this is distinct from a maximal independent set of a closure system.) When $\sigma$ is a closure operator of the corresponding convex geometry, $B = E \setminus \sigma(E - A)$ is the base of $A$.

The set of the successors of a feasible set $A$ in an antimatroid $\mathcal{F}$ is denoted by $\Gamma(A) = \{e \in E \setminus A : A \cup e \in \mathcal{F}\}$. Naturally $\Gamma(A) = \text{ex}(E \setminus A)$ holds.

For a subset $A$ and $X = E - A$, it is obvious that

$$x \in \Gamma(A) \iff A \cup x \in \mathcal{F} \iff E - (A \cup x) \in K \iff X - x \in K \iff \sigma(X - x) = X - x \iff x \not\in \sigma(X - x) = \text{ex}(X).$$
7.3 The characterizations of convex geometries

First recall that the anti-exchange property of a closure operator is below.

\[ x \neq y, \ x \not\in \sigma(A), \ x \in \sigma(A \cup y) \implies y \not\in \sigma(A \cup x) \quad (x, y \in E, A \subseteq E). \]  

(7.4)

The following is an equivalent statement of the anti-exchange property.

\[ x \neq y, \ x \in \sigma(A \cup y), \ y \in \sigma(A \cup x) \implies x, y \in \sigma(A) \quad (x, y \in E, A \subseteq E). \]  

(7.5)

**Proposition 7.4** (7.4) and (7.5) are equivalent.

(Proof) (7.4) \(\implies\) (7.5): Suppose \(x \in \sigma(A \cup y)\) and \(y \in \sigma(A \cup x)\). If \(x \not\in \sigma(A)\), then \(y \not\in \sigma(A \cup x)\) follows from (7.4), a contradiction. Similarly \(y \not\in \sigma(A)\) leads to a contradiction.

(7.5) \(\implies\) (7.4): Suppose that the left-hand side of (7.4) holds. If \(y \in \sigma(A \cup x)\), \(x \in \sigma(A \cup y)\) and \(y \in \sigma(A \cup x)\), then \(x, y \in \sigma(A)\) follows from (7.5), which is a contradiction. Hence either \(y \not\in \sigma(A \cup x)\) or \(x \not\in \sigma(A \cup y)\) must hold.

In contrast, the condition below is called the (Steinitz-McLain) exchange property.

\[ x \neq y, \ x \not\in \sigma(A), \ x \in \sigma(A \cup y) \implies y \in \sigma(A \cup x) \quad (x, y \in E, A \subseteq E) \]  

(7.6)

A closure operator satisfies the exchange property if and only if it is the closure operator of a matroid.

We shall list up the main characterizations of convex geometries in the following.

**Lemma 7.5** For a closure space \((\mathcal{K}, \sigma, E)\) and any subset \(A \subseteq E\), \(\text{ex}(\sigma(A)) \subseteq A\) holds.

(Proof) Suppose \(a \in \sigma(A)\) and \(a \not\in A\). Then \(\sigma(\sigma(A) - a) \supseteq \sigma(A - a) = \text{sigma}(A) \ni a\). Hence \(a \in \sigma(\sigma(A) - a)\) and \(a \not\in \text{ex}(\sigma(A))\). Now the assertion is proved.

**Theorem 7.6** For closure system \((E, \mathcal{K})\), the following are equivalent.

(A) \(\mathcal{K}\) is a convex geometry, i.e. for any \(X \in \mathcal{K}\), \(X \neq E\), there exists an element \(e \in E - X\) such that \(X \cup e \in \mathcal{K}\).

(S) \(\mathcal{K}\) forms a lower semimodular lattice of height \(|E| - |\sigma(\emptyset)|\).

(P) \(\mathcal{K}\) is a graded poset with respect to inclusion relation, and the length of the maximal chains between the minimum \(\sigma(\emptyset)\) and the maximum \(E\) is \(|E| - |\sigma(\emptyset)|\).

(Q) If \(A\) covers \(B\) \((A, B \in \mathcal{K})\), then \(|A| = |B| + 1\).

(Proof) (A) \(\implies\) (S) Assume that \(A \lor B\) covers \(A\) \((A, B \in \mathcal{K})\). Then we wish to show that \(B\) covers \(A \land B = A \lor B\). By (A), there is an increasing chain \(A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_k = E\) in \(\mathcal{K}\) with \(|A_i| = |A_{i-1}| + 1\) for \(i = 1, \ldots, k\), which gives a chain \(A = A_0 \cap A' \subseteq A_1 \cap A' \subseteq \cdots \subseteq A_k \cap A' = A'\). Let \(j\) be the first index such that \(A = A_0 \cap A' = A_j \cap A' \subseteq A_{j+1} \cap A' \subseteq A'\). By assumption, \(A \lor B = A' = A_{j+1} \cap A'\) and \(|A'| = |A| + 1\). Hence \(A' = A \lor B = A \lor e\) for some \(e \in B - A\). Thus \(B = (A \lor B) \lor e\) covers \(A \land B = A \lor B\).

(S) \(\implies\) (P) follows from Lemma 4.14. (P) \(\implies\) (Q) is obvious. Hence (A), (S), (P), (Q) and (T) are equivalent.

In a closure system, a subset \(B\) of \(A\) is a generating set of \(A\) if \(\sigma(A) = \sigma(B)\). Note that a generating set of \(A\) is a generating set of \(\sigma(A)\), but the converse is not true for closure systems in general.
Theorem 7.7 ([59, 61, 63, 94, 112]) Let \((E, \mathcal{K})\) be a closure system. And let \(\sigma\) and \(\text{ex}\) denote the associated closure operator and the extreme-point operator, respectively. Then the following are equivalent

(A) \(\mathcal{K}\) is a convex geometry, i.e. for any \(X \in \mathcal{K}\), \(X \neq E\), there exists an element \(e \in E - X\) such that \(X \cup e \in \mathcal{K}\).

(B) For any subset \(A \subseteq E\), \(\sigma(\text{ex}(A)) = \sigma(A)\).

(B') For any closed set \(X \in \mathcal{K}\), \(\sigma(\text{ex}(X)) = X\), (Krein-Milman property)

(C) For any subset \(A \subseteq E\), \(\text{ex}(A)\) is the unique minimal generating set of \(A\).

(D) \(\text{ex}(\sigma(A)) = \text{ex}(A)\) for any \(A \subseteq E\).

(E) For any closed set \(X \in \mathcal{K}\) and an element \(p \in E - X\), \(p \in \text{ex}(\sigma(X \cup p))\).

(F) The closure operator \(\sigma\) meets the anti-exchange property.

(Proof)
We shall show first \((A) \Rightarrow (B) \Rightarrow (C) \Rightarrow (D) \Rightarrow (E) \Rightarrow (F) \Rightarrow (A)\).

\((A) \Rightarrow (B)\)
Since \(\text{ex}(A) \subseteq A\), \(\sigma(\text{ex}(A)) \subseteq \sigma(A)\). Suppose \(\sigma(\text{ex}(A)) \subsetneq \sigma(A)\) for some \(A \subseteq E\). By the strong accessibility (AC'), there exists a set \(Y \subseteq \sigma(A) - \sigma(\text{ex}(A))\) such that \(Y \cup \sigma(\text{ex}(A))\) is a closed set and \(|\sigma(\text{ex}(A)) \cup Y - \sigma(\text{ex}(A))| = 1\). Let \(\{e_0\} = (\sigma(\text{ex}(A)) \cup Y) - \sigma(\text{ex}(A))\). Then \(e_0 \notin \sigma(\sigma(\text{ex}(A)) \cup Y) = \sigma(A - e_0)\). Hence we have \(e_0 \in \text{ex}(A) \subseteq \sigma(A)\), which is a contradiction. Hence \(\sigma(\text{ex}(A)) = \sigma(A)\).

\((B) \Rightarrow (C)\)
From \((B)\), we have \(\sigma(\text{ex}(A)) = \sigma(A)\). Suppose \(Y \subseteq A\) to be an arbitrary generating set of \(A\). First we prove \(\text{ex}(A) \subseteq Y\). If there exists \(e \in \text{ex}(A)\) with \(e \notin Y\), then \(e \in \sigma(A) = \sigma(Y) = \sigma(Y - e) \subseteq \sigma(A - e)\) as \(Y \subseteq A\). Hence \(e \notin \text{ex}(A)\), which is a contradiction. Thus \(\text{ex}(A) \subseteq Y\) necessarily holds. Now \(\text{ex}(A)\) is shown to be the unique minimal generating set of \(A\).

\((C) \Rightarrow (D)\)
From \((C)\), \(\text{ex}(A)\) is the unique minimal generating set of \(A\), and \(\text{ex}(\sigma(A))\) is the unique minimal generating set. Since a minimal generating set of \(A\) is a minimal generating set of \(\sigma(A)\), they must be equal, namely, \(\text{ex}(A) = \text{ex}(\sigma(A))\).

\((D) \Rightarrow (E)\)
Take any \(X \in \mathcal{K}\) and \(p \in E - X\). Since \(p \notin X = \sigma(X)\), by the definition of extreme-point operators, \(p \in \text{ex}(X \cup p)\). \((D)\) yields \(\text{ex}(X \cup p) = \text{ex}(\sigma(X \cup p))\). Hence \(p \notin \text{ex}(\sigma(X \cup p))\).

\((E) \Rightarrow (F)\)
Suppose \(X \in \mathcal{K}\), \(p, q \notin X\), \(p \neq q\), and \(q \in \sigma(X \cup p)\). From \((E)\), we attain \(p \in \text{ex}(\sigma(X \cup p))\), that is, \(p \notin \sigma(\sigma(X \cup p) - p)\). Since \(q \in \sigma(X \cup p)\), we have \(\sigma(X \cup q) \subseteq \sigma(X \cup p)\). Now if \(p \in \sigma(X \cup q)\), then
\[
\sigma(X \cup q) = \sigma(\sigma(X \cup q)) = \sigma(\sigma(X \cup q) - p) \subseteq \sigma(\sigma(X \cup p) - p).
\]
This implies \(p \in \sigma(\sigma(X \cup p) - p)\), a contradiction. Hence \(p \notin \sigma(X \cup q)\).
(F) ⇒ (A)

Let $X \in \mathcal{K}$ be a closed set with $X \neq E$. Suppose $X \cup y \not\in \mathcal{K}$ for every $y \in E \setminus X$. Then $X \cup y \not\subseteq \sigma(X \cup y)$. Take such an element $a$ that it satisfies these conditions and that $\sigma(X \cup a)$ is minimal.

Since $X \cup a \not\in \mathcal{K}$, there exists $b \in \sigma(X \cup a) \setminus (X \cup a)$. This implies $X \cup b \not\subseteq \sigma(X \cup a)$, so that $b \in \sigma(X \cup b) \subseteq \sigma(X \cup a)$. Now we have $a, b \not\in \sigma(X)$ and $b \in \sigma(X \cup a)$. By the anti-exchange property (F), we have $a \not\in \sigma(X \cup b)$. This implies $\sigma(X \cup b) \not\subseteq \sigma(X \cup a)$, and contradicts the minimality of $\sigma(X \cup a)$.

Lastly, we shall show (B) ⇒ (B') ⇒ (E). It is straightforward that (B) ⇒ (B').

We shall prove (B') ⇒ (E). From Lemma 7.5, $\text{ex}(\sigma(X \cup p)) \subseteq X \cup p$ follows. Suppose contrarily that $p \notin \text{ex}(\sigma(X \cup p))$. Then we have $\text{ex}(\sigma(X \cup p)) \subseteq X$, and $\sigma(\text{ex}(\sigma(X \cup p))) \subseteq \sigma(X) = X$. By the assumption (B'), $\sigma(\text{ex}(\sigma(X \cup p))) = \sigma(X \cup p)$ holds. Hence $p \in \sigma(X \cup p) \subseteq \sigma(X) = X$. This is a contradiction. Thus $p \in \text{ex}(\sigma(X \cup p))$ is shown. \qed

**Proposition 7.8 (Edelman [59])** A finite lattice $L$ is isomorphic to a convex geometry if and only if it is a meet-distributive lattice.

(Proof) Suppose $L$ is isomorphic to a convex geometry $(\mathcal{K}, E)$. For any closed set $X$ in $\mathcal{K}$, assume that $X$ covers $X_1, \ldots, X_k$ in $\mathcal{K}$. By Theorem 7.6 (Q), there are distinct elements $e_1, \ldots, e_k$ such that $X_j = X - e_j$ for $j = 1, \ldots, k$. Since a convex geometry is a closure system, in particular closed with respect to set intersection, it is obvious that the interval $[X_1 \cap \ldots \cap X_k, X]$ is a Boolean lattice. Hence $\mathcal{K}$ is a meet-distributive lattice.

Conversely suppose $L$ is a meet-distributive lattice. By Proposition 5.9, a finite lattice $L$ is isomorphic to a closure system $\mathcal{K} = \{J(x) : x \in L\}$. Now assume $X = J(x)$ covers $Y = J(y)$ in $\mathcal{K}$. By definition, $J(y) \subseteq J(x)$ holds. In other words, $J(x) = \{a_1, \ldots, a_m\}$, $J(y) = \{a_1, \ldots, a_k\}$ and $k < m$. If $k \leq m - 2$, putting $z = a_1 \lor \cdots \lor a_{k+1} \in L$, we have $y < z < x$, and $J(y) \subseteq J(z) \subseteq J(x)$. This contradicts the covering relation between $X$ and $Y$. Hence $k = m - 1$. Then it follows from Theorem 7.6 (Q) that $\mathcal{K} = \{J(x) : x \in L\}$ is a convex geometry. \qed

We remark here that even if a closure system is a meet-distributive lattice, it is not necessarily a convex geometry. Suppose a convex geometry $\mathcal{K}$ is given, take any filter $\emptyset \neq F \subseteq \mathcal{K}$. Let $e'$ a new element, and $\mathcal{K}' = \{X \in \mathcal{K} : X \not\in F\} \cup \{X \cup e' : X \in F\}$. Even though $\mathcal{K}'$ is isomorphic to $\mathcal{K}$ as a lattice, $\mathcal{K}'$ is no more a convex geometry.

Let $(\mathcal{K}, \sigma, E)$ be a convex geometry, and $\mathcal{I} = \text{In}(\mathcal{K})$ be the collection of independent sets with respect to $\tau$. By (C), $\sigma \circ \text{ex} = \text{id}_\mathcal{K}$, and by $E$, $\text{ex} \circ \sigma = \text{id}_\mathcal{I}$. That is, $\sigma$ and $\text{ex}$ are the inverse operator of each other between $\mathcal{K}$ and $\mathcal{I}$. Hence they are isomorphic as a lattice. The union and the meet in $\mathcal{K}$ naturally define a union and a meet in $\mathcal{I}$ as follows. For $A, B \in \mathcal{I}$,

$$A \lor B = \text{ex}((\sigma(A) \cup \sigma(B)))$$
$$A \land B = \text{ex}((\sigma(A) \cap \sigma(B)))$$

Note that the lattice of $\mathcal{I}$ is determined with respect to the above join and meet and not with respect to inclusion relation. Actually, as is seen before, $\mathcal{I}$ is a simplicial complex as a set family.

Convex geometries can be characterized in view of the relation of closure operators and extreme-point operators. We can say that an extreme-point operator is the central concept even to a closure operator.
This equivalence will be well extended to a wider framework of intensive operators and extensive operators in Chapter 14.

Any function \( f : 2^E \to 2^E \) simply gives rise to an equivalence relation \( \equiv_f \) on \( 2^E \); for \( A, B \in 2^E \), \( A \equiv_f B \) if \( f(A) = f(B) \). For a closure space \( (K, \sigma, \text{ex}, E) \), \( K \) is a convex geometry if and only if a pair of equivalence relations \( \equiv_\sigma \) and \( \equiv_{\text{ex}} \) are equivalent. Please see below. Hence a convex geometry has a canonical partition of \( 2^E \) derived from these equivalence relations.

**Theorem 7.9** ([6, 63, 113, 125, 126]) Suppose that \( (K, \sigma, E) \) is a closure space, and \( \text{In}(K) \) is the collection of independent sets of a closure system \( K \). Then the following are equivalent.

1. \( K \) is a convex geometry.
2. \( \sigma \circ \text{ex} \) is the identity map on \( K \),
3. \( \text{ex} \circ \sigma \) is then identity map on \( \text{I}(K) \).
4. Two equivalence relations \( \equiv_\sigma \) and \( \equiv_{\text{ex}} \) are equal.
5. \( \text{ex}|_K : K \to \text{I}(K) \) is a bijection.
6. \( \sigma|_{\text{I}(K)} : \text{I}(K) \to K \) is a bijection.
7. For every independent set \( A \in \text{I}(K) \), \( \text{ex}^{-1}(A) \) is an interval in \( 2^E \).
8. For every closed set \( X \in K \), \( \sigma^{-1}(X) \) is an interval in \( 2^E \).

Given a closure system \( K \), for \( X \in 2^E \), let us define a relation \( \preceq_X \) by \( p \preceq_X q \) for \( p, q \in E - X \). If \( \preceq_X \) is a partial order on \( E - X \) for every closed set \( X \in K \), then \( K \) is a convex geometry. If \( \preceq_X \) is an equivalence relation for every \( X \in 2^E \), then \( K \) is the lattice of closed sets of a matroid.

### 7.4 Rooted circuits of convex geometries

We have defined a rooted circuit for closure systems generally. The idea of rooted circuit stems from the rooted circuits in convex geometries.

**Lemma 7.10** For a convex geometry \( (K, \sigma, E) \), every circuit includes a unique non-extreme element.

(Proof) Let \( C \) be a circuit (i.e. minimal dependent set). Since it is dependent, it contains a non-extreme element \( r \in C \). By definition, \( r \in \sigma(C - r) \). Hence we have \( C = (C - r) \cup r \subseteq \sigma(C - r) \subseteq \sigma(C) \). It follows that \( \sigma(c - r) = \sigma(C) \). By the definition of circuits, \( C - r \) is an independent set, namely, \( \text{ex}(C - r) = C - r \). Then

\[
\text{ex}(\sigma(C)) = \text{ex}(\sigma(C - r)) = \text{ex}(C - r) = C - r.
\]

Thus \( r \) is the unique non-extreme element of \( C \). \( \square \)

In a convex geometry, the unique non-extreme element \( e \) in a circuit \( C \) is called the root and \( C - e \) is called the stem of \( C \). Hence a circuit of a convex geometry is uniquely partitioned into the root \( e \) and the stem \( C \setminus e \). In Dietrich [50] and Korte, Lovász and Schrader [112], the pair \( (C, e) \) is called a rooted circuit. We use different terminology, and call a rooted set \( (C \setminus e, e) \) a rooted circuit of a convex geometry.

We have already defined the rooted circuits for closure systems generally, and we shall show that they coincide with each other in convex geometries.
**Theorem 7.11** Let \((\mathcal{K}, E)\) be a convex geometry. Suppose \(e\) to be an element of \(E\), and \(X\) to be a subset of \(E - e\). Then the following conditions are equivalent.

1. \(X \cup e\) is a circuit, and \(e\) is a non-extreme element of \(X \cup e\).
2. \((X, e)\) is a rooted circuit of the closure system \(\mathcal{K}\).

(Proof) (1) \(\implies\) (2) is already shown in Proposition 5.13.

(2) \(\implies\) (1): Suppose \((X, e)\) to be a rooted circuit, and we shall show that \(X \cup e\) is a circuit. Since \(X \cup e\) includes a non-extreme element \(e\), \(X \cup e\) is a dependent set. We shall show the minimality of \(X \cup e\).

By definition, \(e\) is a non-extreme element in \(X \cup e\). First note that \((X \cup e) - e = X\) is an independent set. If otherwise, there is a non-extreme element \(a\) in \(X\), which implies \(X \subseteq \sigma(X - a)\). This contradicts the minimality of \(X\). Hence \(X\) is an independent set.

Next for any \(f \in X\), \((X - f) \cup e\) is to be proved an independent set. Set \(X' = X - f\). If \(e \notin \text{ex}(X' \cup e)\), then \(e \in \sigma(X')\). This contradicts the minimality of \(X\). Hence \(e\) is an extreme element of \(X' \cup e\). Next we show that every element \(z\) of \(X'\) is an extreme element of \(X' \cup e\). Suppose contrarily that \(z\) is not an extreme element. Then \(z \in \sigma((X' - z) \cup e)\). From the minimality of \(X\), \(e \notin \sigma(X' - z)\) follows. Since \(X'\) is an independent set, \(z \notin \sigma(X' - z)\). By the anti-exchange property, we have \(e \in \sigma((X' - z) \cup z) = \sigma(X')\). This contradicts the minimality of \(X\). Hence \(z\) is an extreme element of \((X - f) \cup e\).

All together with above arguments, every element of \((X - f) \cup e\) is seen to an extreme element. Hence \((X - f) \cup e\) is an independent set.

Now we know that if deleting any element from \(X \cup e\), the resultant set is independent. Hence \(X \cup e\) is known to be a circuit. \(\Box\)

We have also defined a critical rooted circuit for closure systems. We shall show that the definition of critical circuits in [112] and our definition are equivalent in case of convex geometries.

**Proposition 7.12** Let \(\mathcal{K}\) and \(\mathcal{F}\) be a dual pair of a convex geometry and an antimatroid on \(E\), and \(\sigma\) be the associated closure operator of \(\mathcal{K}\). Then for a rooted circuit \((X, a)\), the following are equivalent.

1. \((X, a)\) is a critical rooted circuit.
2. Let \(A = E - \sigma(X) \in \mathcal{F}\) be the maximum feasible set in \(E - (X \cup a)\). Then \(A \cup a \notin \mathcal{F}\), and \(A \cup a \cup b \in \mathcal{F}\) for every \(b \in X\).

(Proof) (1) \(\implies\) (2): Since \(a\) is not an extreme point of \(X \cup a\), \(A \cup a \notin \mathcal{F}\) is obvious. Suppose there exists \(f \in X\) such that \(A \cup f \cup a \notin \mathcal{F}\). Since \(f \in X = \text{ex}(X) = \text{ex}(\sigma(X))\), \(A \cup f \in \mathcal{F}\) holds. Hence \(a \in \sigma(X) - f \notin \mathcal{K}\). Now \(A \cup f \cup a \notin \mathcal{F}\) implies \(a \notin \text{ex}(E - (A \cup f)) = \text{ex}(\sigma(X) - f)\). Thus there exists a rooted circuit \((Y, a)\) such that \(\sigma(Y) \subseteq \sigma(X) - f \subseteq \sigma(X)\). This contradicts the minimality of \(\sigma(X)\) of a critical circuit.

(2) \(\implies\) (1): Suppose conversely that (2) holds. As is described above, \(A \cup b \in \mathcal{F} - a\) for any \(b \in X\). Let \(\mathcal{F}' = \mathcal{F} - a\) and \(\mathcal{K}' = \mathcal{K} - A\). Then by assumption, \(\Gamma_{\mathcal{F}'}(A) = X = \text{ex}_{\mathcal{K}'}(X)\). Hence \(X\) is an independent set in \(\mathcal{K} - a\). By assumption, \(A \in \mathcal{F} - a\), and \(X = (E \setminus a) - A \in \mathcal{K}'\). Thus \(X\) is a closed set in \(\mathcal{K}'\). \(\Box\)

In case of a convex geometry, the collection of the critical circuits is necessary and sufficient to determine the convex geometry (in the sense of Proposition 5.11).
**Proposition 7.13** ([112]) Let $\mathcal{K}$ be a convex geometry, and $C_r$ be the collection of all the critical circuits. Then

1. $\mathcal{K}$ is determined by its critical circuits, namely, $\mathcal{K} = \mathcal{K}[C_r]$.
2. If any collection $C_0$ of rooted circuits determines $\mathcal{K}$, that is, $\mathcal{K} = \mathcal{K}[C_0]$, then $C_0$ contains all the critical circuits.

(Proof)

1. Let $C$ be the set of the rooted circuits of $\mathcal{K}$. In view of Proposition 5.15, $\mathcal{K}$ is determined from $C$, i.e. $\mathcal{K} = \mathcal{K}[C]$. Let $\mathcal{K}_0$ be the closure system defined from $C_0$. Then obviously, $\mathcal{K} \subseteq \mathcal{K}_0$ since $C_0 \subseteq C$. Suppose that there exists $A \in \mathcal{K}_0$ such that $A \not\in \mathcal{K}$. Then there must exist a rooted circuit $(X, e) \in C \setminus C_0$ such that $X \subseteq A$ and $e \not\in A$. Take such $X$ that is minimal. Since $(X, e)$ is not critical, there exists $f \in X$ such that $e \not\in \text{ex}((X - f) \cup e)$, i.e. $e \in \sigma(X - f)$. Hence for some $X' \subseteq X - f$, we have a rooted circuit $(X', e) \in C$, and so $X' \nsubseteq X \subseteq A$. This contradicts the minimality of $X$. Hence $\mathcal{K}_0 = \mathcal{K}$.

2. Let $(X, e)$ be a critical circuit. Let $A = E - \sigma(X) \in \mathcal{F}$ be a feasible set. Since $\sigma(X) \setminus e \not\in \mathcal{K}$, there exists a circuit $(Y, e) \in C_0$ such that $Y \subseteq \sigma(X) \setminus e$ and $e \in \sigma(Y)$. From Proposition 7.12, for any $f \in X$, it holds that $A \cup f \cup e \in \mathcal{F}$. Hence we have $E - (A \cup f \cup e) = \sigma(X) - \{e, f\} \in \mathcal{K}$ and $Y \nsubseteq \sigma(X) - \{e, f\}$. If $f \not\in Y$, it leads to $Y \nsubseteq \sigma(X) - \{e, f\}$, a contradiction. Thus $f \in Y$ holds. Since $f \in X$ is arbitrary, $X \subseteq Y$ follows. From the definition of minimality of circuits, we have $X = Y$. Hence $(X, e) \in C_0$. □

< Axioms of convex geometry: rooted circuits >  (Dietrich [50])

Let $C$ be a collection of rooted sets in $E$.

1. (C1) If $(X_1, e_1), (X_2, e_2) \in C$ and $X_1 \subseteq X_2$, then $X_1 = X_2$.

2. (C2) If $(X_1, e_1), (X_2, e_2) \in C$ and $e_1 \in X_2$, then there exists $X_3 \subseteq (X_1 \cup X_2) \setminus e_1$ such that $(X_3, e_2) \in C$.

As is described in Theorem 5.16, a collection of rooted circuits gives rise to a closure system.

**Theorem 7.14** Suppose that a family $C$ of rooted sets satisfies (C1) and (C2). When we define $\tau : 2^E \rightarrow 2^E$ by

$$
\tau(A) = A \cup \{ e \in E \setminus A \mid \exists X \subseteq A, \ (X, e) \in C \}, \quad (A \subseteq E)
$$

$\tau$ is an anti-exchange closure operator. Furthermore, the collection of rooted circuits of a convex geometry arising from $\tau$ equals $C$.

(Proof) First we shall show $\tau$ is a closure operator. $A \subseteq \tau(A)$ and $A \subseteq B \Rightarrow \tau(A) \subseteq \tau(B)$ are obvious. The idempotency, i.e. $\tau(\tau(A)) = \tau(A)$ for every $A \subseteq E$, is left to prove.

$\tau(A) \subseteq \tau(\tau(A))$ is immediate from the definition. Suppose contrarily that there exists an element $a \in \tau(\tau(A)) - \tau(A)$. By definition, there exists $X_1 \subseteq \tau(A)$ such that $(X_1, a) \in C$. Since $X_1 \subseteq A$, $a$ belongs to $\tau(A)$, which is a contradiction. Hence there is an element $b \in X_1 \setminus A$. But we have $b \not\in A$ and $b \in \tau(A)$. Also by definition, there exists $X_2$ such that $(X_2, b) \in C$. By (C2), there is a rooted circuit $(X_3, b) \in C$ such that $X_3 \subseteq (X_1 \cup X_2) \setminus a \subseteq A$. This implies $b \in \tau(A)$, a contradiction. Thus $\tau$ is shown to be a closure operator.

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We shall show next that there is an element whose stem is included in . Then there must exist an element , which is a contradiction. Thus is shown to meet the anti-exchange property.

Now determines a convex geometry . Let be a family of rooted circuits of . Lastly we shall show .

Let us first prove . Take an arbitrarily . By the definition of rooted circuits, there is a set such that . From the definition of , there exists such that . It is clear that while and hold. Hence from the assumption (C1), we have .

The reverse inclusion is to be proved. Suppose . From the definition of rooted circuits, we have . But our definition of indicates that there is a set with . This readily gives . Because of the minimality of circuits, we have . Hence belongs to . This completes the proof.

The collection of feasible sets of the corresponding antimatroid is given as follows [50].

\[ \mathcal{F} = \{ A \subseteq E : (X,r) \in \mathcal{C} \Rightarrow A \cap (X \cup \{r\}) \neq \{r\}\}. \]

Similarly as is already mentioned in Proposition 5.15, a rooted circuit system determines the corresponding convex geometry.

A subset of is said to be if it does not include a stem of any circuit. By definition, the collection of nbc-independent sets is a simplicial complex, which we call a broken circuit complex and denoted by .

**Theorem 7.15** For a convex geometry , a subset is nbc-independent if and only if it is a free set. Namely, the broken circuit complex and the free complex are equal.

(Proof) For the proof of sufficiency, we suppose to be nbc-independent. If is not a closed set, then there must exist an element . This implies that there exists a circuit for which is its root and its stem is included in . Contradicting our assumption. Hence is proved to be a closed set.

We shall show next that is independent. Suppose that is not independent and . Then there is an element such that . Since , this implies that there exists a subset in which is a stem of a circuit with its root . That is, as well as includes a stem, which is a contradiction again. Hence, we have , and is independent. Now is proved to be a free set.

For the necessity part, is supposed to be free, and we shall show is nbc-independent. Suppose contrarily that contains a stem of a circuit . Let be the root of . In case that and both hold. Hence is not a closed set, but from Lemma 5.4, every subset of a free set is free and is particularly closed. This is a contradiction. On the other hand, in case of , we have , which is a contradiction. Henceforth does not contain any stem, and so is nbc-independent.

Hence the free complex and the broken circuit complex are both subcomplexes of .

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7.5 Rooted cocircuits of convex geometries

First recall that in a finite lattice, any element except 0 is a join of some join-irreducible elements. Let \((\mathcal{A}, E)\) be an antimatroid, and \(\mathcal{J}(\mathcal{A})\) be the collection of join-irreducible elements in the lattice \(\mathcal{A}\). Each element \(Y \in \mathcal{J}(\mathcal{A})\) covers uniquely an element \(Y' \in \mathcal{A}\). Then by the strong accessibility \((\text{AC}')\), \(Y - Y' = \{e\}\) for some \(e \in E\). We call the pair \((Y \setminus e, e)\) a rooted cocircuit.

< Axioms of antimatroids/convex geometries: Rooted cocircuits > ([112])

Let \(\mathcal{D} = \{(Y_i, e_i : i \in I)\}\) be a collection of rooted sets on a finite set \(E\). For every \((Y, e) \in \mathcal{D}\) and \(f \in Y\),

(D1) there exists a rooted set \((Y', f) \in \mathcal{D}\) such that \(Y' \subseteq Y \setminus f\);

(D2) there does not exist a rooted set \((Y', f) \in \mathcal{D}\) with \(e \in Y' \cup f \subseteq Y \cup e\).

**Proposition 7.16** For a set of rooted sets \(\mathcal{D} = \{(Y_i, e_i) : i \in I\}\) satisfying (D1) and (D2),

\[ \mathcal{A}_D = \{\emptyset\} \cup \bigcup \{Y_i \cup e_i : i \in H, H \subseteq I\} \]

is an antimatroid. Moreover, the collection of rooted cocircuits of \(\mathcal{A}_D\) is \(\mathcal{D}\).

(Proof) By definition, \(\mathcal{A}_D\) is closed under union, and contains the empty set.

First we shall show \(Y_i \in \mathcal{A}\) for each \(i \in I\). Fix \(i \in I\), and suppose \(Y_i = \{y_1, \ldots, y_k\}\). For each \(y_j (1 \leq j \leq k)\), by (D2), there exists \((Y'_j, y_j)\) with \(Y'_j \cup y_j \subseteq Y\). Hence we have \(Y = \cup_{j=1, \ldots, k}(Y'_j \cup y_j) \in \mathcal{A}_D\).

Now suppose \(F \in \mathcal{A}\) and \(F \neq \emptyset\). By definition, \(F = \cup_{j \in H}(Y_j \cup e_j)\) for some \(H \subseteq I\), \(H \neq \emptyset\). Take any \(p \in H\). Since \(Y_p \in \mathcal{A}_D\) from the above argument, \(F - e_p = Y_q \cup (\bigcup_{j \in H, j \neq p}(Y'_j \cup e_j)) \in \mathcal{A}_D\) holds, and the accessibility (AC) is satisfied. Hence \(\mathcal{A}_D\) is an antimatroid.

Next suppose \((W, f)\) is a rooted cocircuit of \(\mathcal{A}_D\), namely, \(W \cup f\) is a join-irreducible elements in \(\mathcal{A}_D\) uniquely covering \(W\). By definition, \(W \cup f = \cup_{j \in J}(Y_j \cup e_j)\) for some \(J \subseteq I\). Then clearly there exists uniquely \(j\) such that \(e_j = f\) and \(W = Y_j\).

Conversely we show that each \((Y_i, e_i) \in \mathcal{D}\) is a rooted cocircuit of \(\mathcal{A}_D\). Let \(P_i = Y_i \cup e_i\) for \(i \in I\). Suppose \(P_k = Y_k \cup e_k\) is not a join-irreducible element of \(\mathcal{A}_D\). Then \(P_k = \cup_{j \in J}P_j\) for some \(J \subseteq I\) \(\setminus k\). For every \(j \in J\), \(e_k \notin P_j\) follows from the condition (D2). Hence \(e_k \notin \cup_{j \in J}P_j = P_k\), which is a contradiction. Thus \(P_i = Y_i \cup e_i\) is a join-irreducible element for any \(i \in I\).

Please note here that we do not require the whole underlying set to be necessarily a feasible set of an antimatroid, in other words, the minimum element of a convex geometry to be the empty set. We admit that a convex geometry contains loops.

Summarizing above all, a convex geometry and an antimatroid are just the complement of each other. A convex geometry gives a closure operator, and a closure operator defines the set of rooted circuits. The given set of rooted circuits determines the original convex geometry by Proposition 5.15. The join-irreducible elements of the lattice of an antimatroid forms a collection of rooted cocircuits, and conversely the set of rooted cocircuits reconstruct the original antimatroid.

For a convex geometry \((\mathcal{K}, E)\), let \(\mathcal{C}(\mathcal{K})\) and \(\mathcal{D}(\mathcal{K})\) denote the collection of rooted circuits and rooted cocircuits of \(\mathcal{K}\), respectively. For an element \(e \in E\), let \(\mathcal{C}_e\) be \(\{X : (X, e) \in \mathcal{C}(\mathcal{K})\}\), which we call a stem clutter, and \(\mathcal{D}(e)\) be \(\{Y : (Y, e) \in \mathcal{D}(\mathcal{K})\}\), which we call a coatm clutter. By definition, a stem clutter \(\mathcal{C}(e)\) and a coatm clutter \(\mathcal{D}(e)\) are both clutters on the set \(E \setminus e\).
For a dual pair \((\mathcal{K}, \mathcal{A})\) of a convex geometry and an antimatroid on \(E\), recall that an element in \(\sigma(\emptyset)\) is said to be a \textit{loop}, and an element in \(\text{ex}(E)\) is called a \textit{coloop} where \(\sigma\) and \(\text{ex}\) are the closure operator and the extreme-point operator of \(\mathcal{K}\), respectively. For an arbitrary element \(e \in E\), \(e\) is a coloop if and only if \(\mathcal{D}(e) = \{\emptyset\}\) and \(\mathcal{C}(e) = \emptyset\). Similarly, \(e\) is a loop if and only if \(\mathcal{C}(e) = \{\emptyset\}\) and \(\mathcal{D}(e) = \emptyset\). If \(e\) is neither a loop nor a coloop, then \(\mathcal{D}_e \neq \emptyset\) and \(\mathcal{C}_e \neq \emptyset\).

We remark that a clutter is a collection of subsets of \(E\) in which any element does not properly contain another. For a clutter \(\mathcal{C}\), \(b(\mathcal{C})\) is the blocker of \(\mathcal{C}\), i.e. the collection of minimal subsets of \(E\) intersecting every element of \(\mathcal{C}\).

**Proposition 7.17** ([112]) Let \((\mathcal{K}, \tau, E)\) be a convex geometry, and \((\mathcal{O}, E)\) be its dual antimatroid. Then for any \(e \in E\) and any set \(X \subseteq E - e\), \(e \in \tau(X)\) if and only if \(A \in \mathcal{O}, e \in A\) signify \(X \cap A \neq \emptyset\).

In particular, for any \(e \in E\), \(b(\mathcal{D}_e) = \mathcal{C}_e\) and \(b(\mathcal{C}_e) = \mathcal{D}_e\).

(Proof) \((\Rightarrow)\) Suppose \(A \in \mathcal{O}, e \in A, X \cap A = \emptyset\). Since \(X \subseteq E - A \in \mathcal{K}\), we have \(\tau(X) \subseteq E - A\). Hence \(e \not\in \tau(X)\). It's a contradiction.

\((\Leftarrow)\) Suppose \(e \not\in \tau(X)\). Set \(A = E - \tau(X)\). Then \(e \in A, A \in \mathcal{O}\). Now it follows from \(X \subseteq \tau(X)\) that \(X \cap A = \emptyset\), and the right-hand side does not hold.

Hence for \(X \subseteq E - e\), \(e \in \tau(X)\) holds if and only if \(X\) is a transversal of \(\mathcal{C}_e\), i.e. \(X\) intersects every element of \(\mathcal{C}_e\). By the definition of minimality, \(\mathcal{C}_e\) and \(\mathcal{D}_e\) are the blocker of each other. \(\square\)

Costem clutter \(\mathcal{D}(e)\) and stem clutter \(\mathcal{C}(e)\)

<table>
<thead>
<tr>
<th>(e)</th>
<th>(\mathcal{D}(e))</th>
<th>(\mathcal{C}(e))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\emptyset)</td>
<td>({12})</td>
</tr>
<tr>
<td>2</td>
<td>(\emptyset)</td>
<td>({2})</td>
</tr>
<tr>
<td>3</td>
<td>({1, 2}), ({2})</td>
<td>({12})</td>
</tr>
<tr>
<td>4</td>
<td>({1, 2}, {2, 3})</td>
<td>({2}), ({1, 3})</td>
</tr>
<tr>
<td>5</td>
<td>({2, 3, 4, 5})</td>
<td>({2}), ({3}), ({4}), ({5})</td>
</tr>
</tbody>
</table>

Figure 7.6: An antimatroid on 5 elements

Proposition 7.17 is helpful for hand calculation of counting up all the rooted circuits since join-irreducible elements in the lattice of an antimatroid are rather easy to find out. Actually, it is easy to check that the join-irreducible elements of the antimatroid of Fig. 7.6 are \(\{1\}\), \(\{2\}\), \(\{1, 2\}\), \(\{2, 3\}\), \(\{1, 2, 4\}\), \(\{2, 3, 4\}\), and \(\{2, 3, 4, 5\}\). From these rooted cocircuits, we can instantly fix costems of each fixed root \(e\). Considering the blocker of \(\mathcal{D}(e)\), we can determine the clutter \(\mathcal{C}(e)\) of stems of a root \(e\).

It is trivial that \(\max\{k : \exists A_1, \ldots, A_k \in \mathcal{C}, A_i \cap A_j = \emptyset (i \neq j)\} \leq \min\{|B| : B \in b(\mathcal{C})\}\). As is mentioned before, if the equality holds, we say that \(\mathcal{C}\) packs. The clutters \(\mathcal{C}(e)\) and \(\mathcal{D}(e)\) do not necessarily pack. In the following Chapter 15, we shall describe the special cases such that these clutters pack.

### 7.6 Precedence structures and antimatroids

For each element \(e\) in a nonempty finite set \(E\), suppose that there is given a collection \(\mathcal{H}(e)\) of subsets of \(E\), which we shall call a \textit{precedence structure}. We assume \(\mathcal{H}(\emptyset) = \emptyset\).
There exist two ways of disjunctive and conjunctive approaches to construct an antimatroid from a precedence structure. Each approach can give rise to an arbitrary antimatroid. Hence the precedence structures can be seen as one of the representations of antimatroids.

(1) Disjunctive precedence structures:

Here a feasible word is defined recursively. The empty word $\epsilon$ is a feasible word since $H(\emptyset) = \emptyset$. Suppose that $f_1 \cdots f_k$ is a feasible word. If there exists some $J \in H(f_{k+1})$ such that $J \subseteq \{f_1, \ldots, f_k\}$ and $f_{k+1} \not\in \{f_1, \ldots, f_k\}$, then $f_1 \cdots f_k f_{k+1}$ is a feasible word. Hence a feasible word is necessarily simple. Korte and Lovász [109] defined and called the same approach an alternative precedence structure.

**Proposition 7.18 ([109])** For the collection of feasible words of an disjunctive precedence structure $F = \{\{e_1 \ldots e_m\}: e_1 \ldots e_m \text{ is a feasible word.}\}$ is an antimatroid on $E$.

(Proof) It is immediate that the collection of feasible words satisfies (L0), (L1), and (L2). Hence $F$ is an antimatroid. □

Conversely, every antimatroid arises from a disjunctive precedence structure. There is a canonical way to reconstruct an antimatroid. That is, the disjunctive precedence structure composed of costem clutters of a convex geometry $(K,E)$ gives rise to the corresponding antimatroid.

**Proposition 7.19 ([112])** Let $(F,E)$ be an antimatroid, and $D(e)$ be the set of costems of each element $e \in E$. When defining an disjunctive precedence structure by $H(e) = D(e)$ for $e \in E$, the resultant antimatroid equals to $F$.

(Proof) By definition, for each element $e$ and a costem $Y \in D(e)$, $Y \cup e$ is a minimal element in $\{A : A \in F, e \in A\}$. We use induction on the length $k$ of the words.

When $k = 0$, nothing is to prove. Suppose that $f_1 \cdots f_k$ is a feasible word, and $\{f_1 \cdots f_k\}$ is a feasible set of $F$. For $x \in E - \{f_1, \ldots, f_k\}$, if there exists such $Y \in D(e)$, then $f_1 \cdots f_k x$ is a feasible word and $\{f_1, \ldots, f_k\} \cup (Y \cup x) = \{f_1, \ldots, f_k, x\} \in F$. Hence for every feasible word $\alpha$, $[\alpha]$ is a feasible set of $F$. Conversely, for each feasible set $A \in F$, there is an elementary chain $\emptyset = A_0 \subseteq \cdots \subseteq A_j = A$ giving rise to a simple word $a_1 \cdots a_j$ where $a_i = A_i - A_{i-1}$ ($i = 1, \ldots, j$). We shall show that $a_1 \cdots a_j$ is necessarily a feasible word of disjunctive precedence structure composed of $D(e)$ ($e \in E$). If it is not a feasible word, take the smallest $m$ such that $a_1 \cdots a_m$ is not a feasible word. But by definition, $A_m = \{a_1, \ldots, a_m\}$ is a feasible set. Hence $A_m$ contains a minimal set $Y$ including $a_m$, i.e. a costem $Y$ in $D(a_m)$. Hence $a_1 \cdots a_m$ is a feasible word. This is a contradiction. □

(2) Conjunctive precedence structure:

We consider a conjunctive style instead of a disjunctive style of alternative precedence structures. Similarly, suppose that a collection $H(e)$ of subsets of $E$ is given for each $e \in E$. 

60
The feasible words are defined as follows. The empty word $e$ is a feasible word, and we take the admission rule so that $f_1 \cdots f_k f_{k+1}$ is feasible if $\{f_1, \ldots, f_k\}$ $(k \geq 1)$ intersects every member of $\mathcal{H}(f_{k+1})$.

**Proposition 7.20** The collection of feasible words of a conjunctive precedence structure is an antimatroid language on $E$.

(Proof) Since an empty word is feasible, $\emptyset \in \mathcal{F}$. Now it is sufficient to prove that if $f_1 \cdots f_k a$ and $f_1 \cdots f_k b$ are feasible words, then $f_1 \cdots f_k ab$ is a feasible word. This is trivial from the definition. □

As is similar to the above, every antimatroid is given by a conjunctive precedence structure. Actually, the stem clutters constitute a standard conjunctive precedence structure.

**Proposition 7.21** Let $(\mathcal{F}, E)$ be an antimatroid. Take a stem clutter $C(e)$ as a precedence structure for each element $e \in E$. Then this conjunctive precedence structure gives rise to the original antimatroid $\mathcal{F}$.

(Proof) Let $C(e)$ be the stem clutter of an element $e$. By proposition 5.15,

$A \in \mathcal{F} \iff B = E \setminus A \in \mathcal{K}$

$\iff$ For every rooted circuit $(X, e)$, either $X \nsubseteq B$ or $e \in B$

$\iff$ For every rooted circuit $(X, e)$, either $X \cap A = \emptyset$ or $e \notin A$.

We use induction on the size $k = |A|$. In case of $k = 0$, the assertion is trivial. Suppose $k > 1$, and let $A_{k-1} = \{e_1, \ldots, e_{k-1}\} \in \mathcal{F}$. From the above argument, $A_k = \{e_1, \ldots, e_{k-1}, e\} \in \mathcal{F}$ for an element $e$ if and only if $X \cap A_{k-1} = \emptyset$ for every $X \in C(e)$. This completes the proof. □

### 7.7 Atomistic convex geometries and 2-graphs

We say a convex geometry is *atomistic* if $\{x\}$ is a closed set for every element $x \in E$. An atomistic convex geometry is automatically loop-free. An antimatroids are called atomistic if the dual convex geometries is atomistic.

A convex geometry is *normal* if $\text{ex}(X) \geq 2$ for any closed set $X \in \mathcal{K}$ with $|X| \geq 2$. Dually, an antimatroid is *normal* if $|\Gamma(A)| = |\text{ex}(E - A)| \geq 2$ for any feasible set $A \in \mathcal{F}$ with $|E \setminus A| \geq 2$.

**Proposition 7.22** A convex geometry is normal if and only if it is atomistic.

(Proof) $\text{アンチマトロイドが正則であるとする。}|E|\text{に関する帰納法で示す。}|E| \leq 1$ ならば自明。$|E| \geq 2$ とする。任意に元 $x \in E$ をとる。仮定から $\Gamma(\emptyset) \geq 2$ なので $x$ とは異なる元 $y$ があって $y \in \Gamma(\emptyset)$ となる。$
\mathcal{F}' = \mathcal{F} - y$ に対して帰納法の仮定から $x$ を最後に持つfeasible sequence $a_1 \cdots a_{n-2} x \in \Gamma(\mathcal{F}')$ が存在する。すなわち, $y a_1 \cdots a_{n-2} x \in \Gamma(\mathcal{F})$ がfeasible sequence なので $E \setminus x \in \mathcal{F}$ と分かった。逆に, *atomistic* 凸幾何が正則であることを示す。凸集合 $X \in \mathcal{K}$ が $|X| \geq 2$ かつ $X$ は join-irreducible とする。定義からある元 $e$ があって $X \setminus e \in \mathcal{K}$ である。単純という仮定から $\{e\} \in \mathcal{K}$ は凸集合。strong accessibility (A1') から $\{e\}$ と $X$ を結ぶ初等鎖 $\{X_1\}_{i=1}^k, X_1 = \{e\}, X_k = X$ が存在する。このとき、$X \not= X_{k-1}, X'$. $X' \vee X_{k-1} \supseteq X' \cup X_{k-1} = X$ かつ $X' \vee X_{k-1} \subseteq X \cup X = X$ である。ゆえに, $X' \vee X_{k-1} = X$.}
We shall denote by $\mathcal{K}^{(k)}$ the set $\{X \in \mathcal{K} : |X| = k\}$. We shall define a graph $G = (E, A)$, called a 2–graph, such that the vertex set is $E = \mathcal{K}^{(1)}$ and the edge set is $A = \mathcal{K}^{(2)}$.

**Lemma 7.23** If $(E, \mathcal{K})$ is an atomistic convex geometry, its 2–graph is a connected graph.

(Proof of Lemma 7.23) $G = (E, A)$ を atomistic 凸幾何 $(E, V)$ の 2–グラフとする。$G$ が非連結として、$E_1 \subseteq E$ を連結成分のひとつで、$E_2 = E \setminus E_1$ とおく。任意の $x \in E_1$ をとる。$\mathcal{K}$ が単純であるから、$\{x\} \in \mathcal{K}$ である。$\{y\}$ から $E$ への初等鎖が存在する。その初等鎖の中で最初に $E_1$ 以外の要素を含む元を $X \in \mathcal{K}$ とする。つまり、$X \cap E_2 = \{y\}$ である。ここで $|X \cap E_2| = 1$ ならば $X = \{x, y\}$ が $E_1$ と $E_2$ を結ぶ $G$ 中の辺になるので、矛盾。

そこで $|X \cap E_1| \geq 2$ とする。ここで $\{y\}$ から $X$ へ至る初等鎖が存在するが、これからただちに $X' = \{x', y\} \in \mathcal{K}$ となる $x' \in E_1$ が存在すると分かる。しかし、$X'$ は $G$ 中で $E_1$ と $E_2$ を結ぶ辺になり、矛盾。□

Now we shall show a 2–graph of a chordal convex geometry of a chordal graph $G$ equals to $G$ if $G$ is connected. In general if a graph $G = (V, A)$ is not connected, we transform $G$ to connected graph $\tilde{G}$ as follows. Let $V = V_1 \cup \cdots \cup V_k$ be the partition into connected components. Then we attach edges and obtain a edge set $A'$ below.

$$A' = A \cup \{uv : u \in V_i, v \in V_j, i \neq j\}$$

The resultant graph $\tilde{G} = (V, A')$ is necessarily a connected graph. If $G$ is a chordal graph, $\tilde{G}$ is naturally chordal.

**Theorem 7.24** Let $G = (V, A)$ be a chordal graph. Then the 2–graph of the chordal convex geometry of $G$ is equal to $\tilde{G}$. Particularly if $G$ is a connected chordal graph, the 2–graph of the chordal convex geometry equals to $G$.

(Proof) $\tilde{G} = (E, \tilde{A})$ とする。$G' = (E, A')$ を $\mathcal{K}$ の 2–グラフとする。

$x y \in A'$ とする。$x$ と $y$ が $G$ の異なる連結成分に属するときは、定義から $x y \in \tilde{A}$ である。$x, y$ が $G$ の辺であるとする。辺 $x y$ を含む成分を $G_0 = (E_0, A_0)$ として、$G_0$ の単体的シェリングのアンチマトロイド $\mathcal{F}_0$ で $E_0 \setminus \{x, y\} \in \mathcal{F}_0$ であることを示せばよい。これは、$x$ を最後にする単体的シェリングの列の中に $x y$ を最後の 2 項にするものが必ず存在することを示せばよい。逆に、そのような列が存在しないとする。その列を $x_1 \cdots x_{k-1} x_k (= y) x_{k+1} \cdots x_n (= x)$ とする。そしてその中で $k$ が最大のものを選ぶ。$G_0$ から $x_1 \cdots x_{k-1}$ を除去したグラフを $G_y$ すると、Lemma 3.4 から $x$ 以外に $x$ と隣接しない単体的点 $y'$ が存在するが、仮定から $y' \neq y$ である。ゆえに $y'$ が $G_0$ で単体的点として除去できるので、これは $k$ を最大に取ることに矛盾する。□

Hence $\mathcal{K}$ が連結な三角化グラフ $G$ の単体的シェリングの凸幾何であれば、$G = \mathcal{K}^{(2)}$ が成立して、$\mathcal{K} = S(\mathcal{K}^{(2)})$ である。

Note that in general if a chordal graph $G$ is not connected, the convex geometry of $G$ and that of $\tilde{G}$ are not equal. Furthermore a 2–graph of a convex geometry is not necessarily a chordal graph.

単純でない凸幾何でも 2–グラフを同様に定義することにすると、一般に連結になるとは限らない。その例。凸幾何の 2–グラフは、一般に三角化グラフになるとは限らない。その例。
\[ E = \{1, 2, 3, 4\}. \text{2}\text{-}E \text{ 中で } \{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\} \text{ で生成されるフィルターを } F_2 \text{ として } F_2 \text{ に関して } 2^E \text{ をliftingしてできた } E' = \{1, 2, 3, 4, 5\} \text{ 上のアンチマトロイドの凸幾何の2−グラフは、三角化グラフになりない。} \]

Example 7.25 2−グラフを与えてもそれを 2−グラフとして凸幾何は一意には定まらない。その例。

予想 G を連結な三角化グラフとして、G を 2−グラフとしても凸 几何を考えると、その中で最小なものと G の単体的凸幾何が一致すると言えるか？

7.8 Path-property of convex geometries

A convex geometry \((\mathcal{K}, \sigma, E)\) has the path-property if it satisfies the following condition: for any closed set \(U \in \mathcal{K}\) with \(|U| \geq 2\) and an element \(x \in U - \text{ex}(U)\), there exist \(y, z \in \text{ex}(U)\), \(y \neq z\) such that \(x \in \sigma(\{y, z\})\).

Proposition 7.26 Let \((\mathcal{K}, E)\) be a convex geometry with \(|E| \geq 2\). Then a convex geometry \(\mathcal{K}\) has the path property if and only if the sizes of the stems of \(\mathcal{K}\) are all two.

(Proof) (if part) Suppose that \(X \in \mathcal{K}\) with \(|X| \geq 2\). If there exists an element \(x \in X - \text{ex}(X)\), then \(X\) is a dependent set, and there exists a rooted circuit \((X', e)\) with \(X' \subseteq X - e\). By assumption, we have \(|X'| = 2\) and we can assume \(X' = \{a, b\}\). These imply the path-property.

(only if part) Let \((X, x)\) be an arbitrary rooted circuit. Then \(x \in \sigma(X) = W \in \mathcal{K}\). By assumption, there exist two distinct elements \(y, z \in \text{ex}(W) = \text{ex}(X)\) such that \(x \in \sigma(\{y, z\})\). From the minimality of rooted circuits, we have \(X = \{y, z\}\). Hence the assertion follows. \(\square\)

Corollary 7.27 A convex geometry satisfying the path-property is atomistic.

(Proof) It is obvious from Propositions 7.22 and 7.26. \(\square\)

Corollary 7.28 If a convex geometry \((\mathcal{K}, E)\) satisfies the path property, then the collection of rooted circuit \(\mathcal{C}\) is as follows.

\[
\mathcal{C} = \{ (\{u, v\}, w) : u, v, w \in E, u, v, w \text{ are distinct, } w \in \sigma(\{u, v\}), u, v \in \text{ex}\sigma(\{u, v\}) \}.
\]

(Proof) Obvious from the definition of rooted circuits. \(\square\)

7.9 2-sum of convex geometries

Following the construction of 2-sum of matroids in Proposition 6.9, we can define an analogous merging of convex geometries.

Let \(C_1, C_2\) be the rooted circuit systems of convex geometries \(K_1, K_2\) on \(E_1, E_2\), respectively. Further suppose \(E_1 \cap E_2 = \{z\}\) and \(|E_1|, |E_2| \geq 3\). Let us define

\[
C(K_1 \Delta K_2) = C_1^* \cup C_2^* \cup C_{1,2} \cup C_{2,1}
\]  

(7.8)
where

\[ C_1^* = \{ (X_1, e_1) : z \not\in X_1 \cup e_1 \} \]
\[ C_2^* = \{ (X_2, e_2) : z \not\in X_2 \cup e_2 \} \]
\[ C_{1,2} = \{ (X_1 \cup (X_2 - z), e_2) : (X_1, z) \in C_1, z \in X_2, (X_2, e_2) \in C_2 \} \]
\[ C_{2,1} = \{ ((X_1 - z) \cup X_2, e_1) : (X_2, z) \in C_2, z \in X_1, (X_1, e_1) \in C_1 \} \]

**Proposition 7.29** $C(K_1 \Delta K_2)$ forms a system of rooted circuits of a convex geometry on $E_1 \Delta E_2$.

(Proof) We shall call a convex geometry defined from $C(K_1 \Delta K_2)$ a 2-sum of $K_1$ and $K_2$. A 2-sum of convex geometries is 2-separable since $K_1 \Delta K_2 - z$ is disconnected.

**Example 7.30** Suppose $H_1$ and $H_2$ be an affine subspaces of dimension $n_1$ and $n_2$ in an $n$-dimensional affine space $\mathbb{R}^n$, respectively, which do not include each other. Further suppose $H_1$ and $H_2$ intersects at a point $z$. Let $E_1$ and $E_2$ be affine point configurations on $H_1$ and $H_2$, and $E_1 \cap E_2 = \{ z \}$. Let $\mathcal{K}$ be the affine convex geometry given by a point configuration $E = E_1 \cup E_2$. Then $\mathcal{K}$ equals a 2-sum $K_1 \Delta K_2$ of convex geometries $K_1$ and $K_2$ where $K_1$ and $K_2$ are the affine convex geometries of $E_1$ and $E_2$, respectively.

### 7.10 Largest extensions of convex geometries

For two closure ststems $\mathcal{K}_1, \mathcal{K}_2 \subseteq 2^E$, we define a partial order $\mathcal{K}_1 \leq \mathcal{K}_2$ if $\mathcal{K}_1$ is a sublattice of $\mathcal{K}_2$ as a lattice. In that case, we call $\mathcal{K}_2$ is an extension of $\mathcal{K}_1$.

We can restate Theorem 4.21 and Corollary 4.22 in terms of convex geometries.

**Theorem 7.31** (Adaricheva et al. [1]) Let $L$ be a finite join-irreducible lattice, and $J(L)$ be the set of join-irreducible elements of $L$. Then $L$ can be embedded to some finite atomistic convex geometry on the underlying set $J(L)$ as a lattice.

**Corollary 7.32** ([1]) A finite atomistic join-semidistributive lattice is isomorphic to a finite atomistic convex geometry.

**Proposition 7.33** ([1]) Every loop-free finite convex geometry has the largest extenion, and the largest extension is an atomistic convex geometry.

(Proof) Let $\mathcal{K}$ be a loop-free convex geometry on a finite set $E$. Let $\mathcal{K}_1, \ldots, \mathcal{K}_m$ be all the extensions of $\mathcal{K}$ on $E$. By Theorem 7.2, $\mathcal{K} = \mathcal{K}_1 \vee \mathcal{K}_2 \vee \cdots \vee \mathcal{K}_m$ is a convex geometry, and is the largest extension of $\mathcal{K}$. By Theorem 7.31, there exists an atomistic convex geometry among $\mathcal{K}_1, \ldots, \mathcal{K}_m$. Hence $\mathcal{K}$ is atomistic. □
Chapter 8

Examples and Classifications of Antimatroids and Convex Geometries

8.1 Major classes of convex geometries and antimatroids

Let us first show four main examples of convex geometries and antimatroids.

(1) Poset convex geometries:

Let $P = (E, \leq)$ be a finite poset. The collection of filters of $P$ is trivially a convex geometry since it is a distributive lattice of height $|E|$. We call it a *poset convex geometry*. Dually, a poset antimatroid is the collection of ideals of a poset $P$. The shelling process of a poset antimatroid is the deletion of a minimal element.

The set of rooted circuits is

$$C = \{\{y\}, x : x, y \in E, y < x\}$$

and the set of rooted cocircuits is

$$D = \{(\emptyset, x) : x \in E \text{ is a minimal element}\} \cup \{(I(x) \setminus x, x) : x \in E \text{ is not a minimal element}\}$$

where $I(x)$ is a principal ideal of an element $x$.

(2) Double shelling convex geometries of posets:

In a poset $P = (E, \leq)$, a subset $A \subseteq E$ is order-convex imply $a, b \in A$ and $a \leq c \leq b$ imply $c \in A$.

The collection of all the order convex sets forms a convex geometry, which is called a *double shelling convex geometry* of a poset ([59]).

The dual of a double shelling convex geometry of a poset is called a *double shelling antimatroid*. The shelling process of this antimatroid is the deletion of a minimal or maximal element. The set of rooted circuits is

$$C = \{\{y, z\}, x : x, y, z \in E, y < x < z\}$$
and the set of rooted cocircuits is

\[ \mathcal{D} = \{ (\emptyset, x) : x \in E \text{ is either minimal or maximal} \} \]
\[ \cup \{ (I(x) \setminus x, x) : x \in E \text{ is neither minimal nor maximal} \} \]
\[ \cup \{ (F(x) \setminus x, x) : x \in E \text{ is neither minimal nor maximal} \} \]

where \( I(x) \) and \( F(x) \) are a principal ideal and a principal filter of the element \( x \), respectively.

**Example 8.1** Fig. 8.1 is a Hasse diagram of a poset \( P \). Fig. 8.2 shows the antimatroid of double shelling of \( P \).

(3) Affine convex geometries:

A finite set \( A \) in an affine space \( \mathbb{R}^n \) is called an affine point configuration in \( \mathbb{R}^n \). For \( X \subseteq A \), \( \text{conv.hull}(X) \) is the ordinary convex hull of \( X \) in \( \mathbb{R}^n \).

\[ \sigma(X) = \text{conv.hull}(X) \cap A \quad (X \subseteq A). \]

is a closure operator on \( 2^A \) satisfying the anti-exchange property. Hence it defines a convex geometry, which we call an affine convex geometry. (Edelman [61])

In this case, the shelling process of an antimatroid is the elimination of an extremea vertex of the convex-hull polytope of the remaining elements.

(4) Chordal convex geometries:

A vertex subset \( X \subseteq V(G) \) is said to be monophonically convex if any vertex on a chordless path connecting a pair of vertices in \( X \) is contained in \( X \).

**Theorem 8.2** (Edelman and Jamison [63]) The collection of monophonically convex sets of a graph \( G \) forms a convex geometry if and only if \( G \) is a chordal graph.

As is seen in Theorem 3.5, if a graph is chordal, there is a sequence of repetitions of deleting a simplicial vertex which deletes all the vertices. Since this simplicial shelling satisfies the properties of a shelling process, this shelling gives rise to an antimatroid, which we shall call a chordal antimatroid.
8.2 Three classes of convex geometries and antimatroids

Fig. 8.3 shows three classes of convex geometries, arising from simplicial shelling of chordal graphs, affine point configurations, and double shelling of posets. Each intersection area is depicted by the letter from a to g.

The following are the examples of convex geometries each of which is contained in the indicated area.

(a) A tree vertex shelling convex geometry of the tree of Fig. 8.4.
(b) A chordal convex geometry of a chordal graph of Fig. 8.5.
(c) An affine convex geometry of point configuration of Fig. 8.6.
(d) A double shelling of a poset of $N_5$ of Fig. 8.7.
(e) A double shelling convex geometry of a poset of Fig. 8.8.
(f) There is no convex geometry of type (f).
(g) A finite Boolean lattice.
(h) A convex geometry of Fig. 8.12 does not belong to any class of chordal convex geometry, poset double shelling and affine convex geometry.

8.3 Affine convex geometries

(1) Affine convex geometries: (See p.66.)
Figure 8.4: (a) Tree

Figure 8.5: (a) Chordal graph

Figure 8.6: (b) Affine point configuration

Figure 8.7: (c) Poset $N_5$

Figure 8.8: (e) Poset

Figure 8.9: (e) Chordal graph

Figure 8.10: (d) Six points configuration in $\mathbb{R}^2$

Figure 8.11: (d) A Hasse diagram of a poset

Figure 8.12: (h) A convex geometry
(2) Semi-lower affine convex geometries (Goecke et al. [79]):

Let $A$ be a finite set and $C$ a *apex cone* in $\mathbb{R}^n$. A cone $C \subseteq \mathbb{R}^n$ is a set such that $x + y \in C$ for every $x, y \in C$ and $kx \in C$ for every $x \in C$ and $k \geq 0$. A cone is *apex* if it has at least one extreme point. The closure operator $\sigma(X)$ for $X \subseteq A$ can be defined by $\sigma(X) = \{x \in A : x \in \text{conv.hull}(X) + C\}$ where $\text{conv.hull}(X) + C = \{x + y : x \in \text{conv.hull}(X), \ y \in C\}$. This is an anti-exchange closure operator, and gives rise to a convex geometry, called a *semi-lower affine convex geometry*. The class of affine convex geometries is a proper subclass of semi-lower affine convex geometries includes.

(3) Acyclic oriented matroids $\mathcal{M} = (C, E)$:

For an acyclic oriented matroid $\mathcal{M}$,

$$\sigma_{\mathcal{M}}(A) = A \cup \{e \in E \setminus A \mid X^+ \subseteq A, X^- = \{e\} \text{ for some } (X^+, X^-) \in \mathcal{M}\}$$

(8.1)

is a closure operator with the anti-exchange property, and hence determines a convex geometry $K_\mathcal{M}$ on $E$ (Edelman [60]).

We shall show some examples of convex geometries arising from acyclic oriented matroids.

(i) An acyclic digraph $G$ gives rise to an acyclic oriented matroid $\mathcal{M}_G$, and hence gives a convex geometry $K_{\mathcal{M}_G}$.

Particularly, if $G$ is transitively closed acyclic digraph, then the convex geometry $K_{\mathcal{M}_G}$ is the transitivity convex geometry of $G$.

(ii) Suppose that an acyclic oriented matroid is affine realizable by a point configuration $E \subseteq \mathbb{R}^n$. Then the convex geometry arising from this acyclic oriented matroid is isomorphic to an affine convex geometry defined on $E$. Hence the class of affine convex geometries is a subclass of those of acyclic oriented matroid.

Fig. 8.13 shows the inclusion relation of the classes of convex geometries.

\[
\begin{array}{c}
\text{acyclic oriented matroids} \\
\downarrow
\end{array}
\begin{array}{c}
\text{affine convex geometries} \\
\downarrow
\end{array}
\begin{array}{c}
\text{acyclic digraphs} \\
\downarrow
\end{array}
\begin{array}{c}
\text{transitively closed acyclic digraphs} \\
\downarrow
\end{array}
\begin{array}{c}
(\text{transitivity convex geometries})
\end{array}
\]

Figure 8.13: Classes of acyclic oriented matroid convex geometries

8.4 Posets, semilattices and lattices

(1) Poset convex geometries:
(i) Poset convex geometries: (See p.1.)

We can characterize poset convex geometries in terms of closure operator or rooted circuit.

**Proposition 8.3 ([112])** For a convex geometry \((K, \sigma, E)\), the following are equivalent.

(a) \(K\) is a poset convex geometry,

(b) \(\sigma(A \cup B) = \sigma(A) \cup \sigma(B)\) for any \(A, B \subseteq E\).

(c) The sizes of stems of rooted circuits are all one,

(Proof) (i)\(\Rightarrow\)(ii): (i) implies that \(K\) is the collection of the filters of a poset. Since the collection of the filters is closed under union, (ii) follows.

(ii)\(\Rightarrow\)(i): (ii) implies that \(K\) is the collection of set closed under union and intersection. By Theorem 4.9, \(K\) is the collection of filters of a poset. Hence \(K\) is a poset convex geometry.

(i)\(\Rightarrow\)(iii): For each \(y \in E\), \(\sigma(\{y\}) = \{x \in E : y \leq x\}\). Hence for a fixed root \(x \in E\), the rooted circuit is of the form \(\{y\}, x\) such that \(y \leq x\) and \(x \neq y\). Thus (3) follows.

(iii)\(\Rightarrow\)(i): As is shown in Theorem 5.15, a convex geometry \(K\) is determined from the family of rooted circuit, say \(C = \{(\{f_i\}, e_i) : i = 1, \ldots, m\}\).

We shall define a directed graph \(H\) such that \(E\) is the vertex set and \(E(H) = \{(f_i, e_i) : i = 1, \ldots, m\}\) is the edge set. \(H\) satisfies transitivity. Actually, if \((a, b), (b, c) \in E(H)\), then (C2) of the circuit axioms of convex geometries implies \((\{a\}, c) \in C\), i.e. \((a, c) \in E(H)\). Next we shall show that \(H\) is acyclic. Suppose \((a_1, a_2, \ldots, a_k, a_0)\) is a directed circuit in \(H\). By definition, \((\{a_k\}, a_0) \in C\). By the transitivity, \((a_0, a_k) \in E(H)\) holds. By the axiom (C2), \((\{a_0\}, a_0)\) is in \(C\). This is a contradiction. Hence \(H\) is a transitively closed acyclic digraph, and hence defines a partial order on \(E\). Now it is obvious that the set of rooted circuits of the poset \(E\) is equal to the original rooted circuit system \(C\). Hence \(K\) is the poset convex geometry of \(E\). □

The rooted circuits and the rooted cocircuits of a poset convex geometry are easy to see. The collections of the rooted circuits and the rooted cocircuits are

\[ C = \{\{y\}, x : y < x\}, \quad D = \{\{w\}, z : z < w\}, \]

respectively.

(ii) Separating sets of a digraph:

Let \(G\) be a digraph, and \(T\) be a subset of \(V(G)\). When a vertex set \(X\) separates \(Y\) from \(T\), we write \(X \preceq Y\). This order \(\preceq\) is in general just a quasi-order. Namely the antisymmetric property does not necessarily hold. For instance, if \(T \subseteq X\), then both of \(X \preceq T\) and \(T \preceq X\) hold. But note that \(X \subseteq Y\) always implies \(X \preceq Y\).

Now let us define

\[ [X] = \{x \in X : \text{there exists a path from } x \text{ to } T \text{ not containing the vertices in } X \text{ other than } x\}. \]

Then \(\{X \subseteq V : X = [X]\}\) is a poset with respect to the partial order \(\preceq\) (Pym and Perfect [143]). Furthermore, it is a meet-distributive lattice, and gives rise to a poset convex geometry (Polat [141], Edelman [61]).

(iii) Barnard convex sets:

Let \(n_1\) and \(n_2\) be two non-negative integers, and \([n_i] = \{0, 1, \ldots, n_i\}\) for \(i = 1, 2\). Set \(X = \)
A subset $A \subseteq X$ is said to be a Barnard convex set if it complies with the following two properties:

(a) if $(i, j) \in A$, then $(i - 1, j) \in A$ for any $1 \leq i \leq n_1$ and $1 \leq j \leq n_2$.

(b) if $(i, j) \in A$, then $(i, j + 1) \in A$ for any $0 \leq i \leq n_1$ and $0 \leq j \leq n_2 - 1$.

Clearly, the collection of Barnard convex sets constitutes a closure system on $X$. Furthermore, let $\sigma(A)$ for $A \subseteq X$ denote the minimum Barnard convex set containing $A$. Then $\sigma$ is the anti-exchange closure operator on $X$, and hence the collection of Barnard convex sets forms a convex geometry (Roy and Stell [145]). Furthermore, it is a poset convex geometry with respect to the partial order

$$(i, j) \leq (i', j')$$

if and only if $i \leq i'$ and $j \geq j'$.

This concept is introduced for the inferiority test in medical statistics.

(2) Double shelling convex geometries of posets: (See p.65)

(3) Transitivity antimatroid (Korte and Lovász [111]) :

Let $G = (V, E) \ (E \subseteq V \times V)$ be a transitively closed acyclic digraph.

Let $G|A$ denote a subgraph induced from an edge set $A \subseteq E$. Then $K = \{A \subseteq E : G|A$ is transitively closed$\}$ forms a convex geometry on $E$, which we shall call a transitivity convex geometry. The dual antimatroid arises from a shelling process of deleting an edge keeping the remaining graph to be transitively closed. More precisely, an edge $x = (a, b) \in E$ is removable if and only if there is no directed path from $a$ to $b$ in $G - x$. The dual of a transitivity antimatroid is precisely a transitivity convex geometry, and the converse is also true.

A rooted circuit of a transitivity antimatroid is simple to state in this case. For an edge $e = (a, b) \in E$, the set of rooted circuits with root $e$ is

$$\mathcal{C}(e) = \{(\{f_1, \ldots, f_k\}, e) : f_1, \ldots, f_k \in E, f_1 \cdots f_k \text{ is a directed path from } a \text{ to } b \text{ in } G - e.\}$$

Furthermore if $e = (u, v) \in E, f = (u, w) \in E$ and $g = (w, v) \in E$, then $((\{f, g\}, e)$ is a critical rooted circuit, and every critical rooted circuit takes this form.

Similarly, a rooted cocircuit is easy to see. The set of rooted cocircuits with respect to an edge $e = (a, b)$ is a minimal directed cut-set separating $a$ and $b$ in $G - e$.

(4) Lattices of partial orders on a set (Edelman and Klinsberg [64], Edelman [60]):

As is stated in Section 3.1, for a partial order $R$ on a finite set $E$, $(E, R)$ is a transitively closed acyclic graph on a finite vertex set $E$, and conversely the set of edges of a transitively closed acyclic graph forms a partial order on $E$.

Convex geometries : Let $Po(E)$ be the collection of all the partial orders on $E$. $Po(E)$ is a poset with respect to inclusion relation. More precisely,

$$P \leq Q \iff P \subseteq Q \quad (\iff \text{for } a, b \in E, \text{ if } a \leq_P b \text{ then } a \leq_Q b ) \quad (8.2)$$
The maximal elements of $Po(E)$ are the total orders. For a fixed partial order $P$, $I(P) = \{Q \in Po(E) : Q \leq P\}$ is the principal ideal in $Po(E)$. It is a meet-distributive lattice, and if $Q$ covers $Q'$ in $I(P)$, then $|Q| = |Q'| + 1$. Hence $I(P)$ is a convex geometry.

As is already mentioned, the transitivity convex geometries of a digraph $(E, P)$ is equal to the convex geometry $I(P)$, and the converse holds. Hence

**Proposition 8.4** The class of the transitivity convex geometries on a finite set $E$ is equal to that of the convex geometries $I(P)$ of the partial orders $P$ on $E$.

(5) The collection of partial orders compatible with a total order:
This is a special case of the above. We take a totally ordered set $N = \{1 < 2 < \cdots < n\}$. As seen above, $I(N)$ is a convex geometry, and forms a meet-distributive lattice. Furthermore, $I(N)$ is a supersolvable lattice (Stanley 1972 [150]).

**Lemma 8.5** $I(N)$ is an affine convex geometry.

(Proof) Let $G$ be a transitively closed digraph corresponding to the totally ordered set $N$. That is, the vertex set $V(G)$ is $\{v_1, v_2, \ldots, v_N\}$ and the edge set is $E(G) = \{(v_i, v_j) : 0 \leq i < j \leq N\}$. $G$ is a transitively closed acyclic digraph, and the transitivity convex geometry of $G$ is equal to $I(N)$. Let $\Delta_N$ be an $(N - 1)$-dimensional simplex in $\mathbb{R}^{N-1}$. Put labels to the vertices of $\Delta_N$ from 1 to $N$. For each edge $(v_i, v_j)$ with $i + 1 < j$, we put a point at the center of the face $\{v_{i+1}, \ldots, v_{j-1}\}$. This definition gives a one-to-one correspondence between the union of $V(G)$ and $E(G)$ and the set of points of $\Delta_N$. The extreme points of $\Delta_N$ correspond to the elements of $N$. Now the affine convex geometry of $\Delta_N$ is equal to the convex geometry $I(N)$. (See Fig.8.14 and Fig. 8.15.)

**Theorem 8.6** The class of transitivity convex geometries is a proper subclass of affine convex geometries.

(Proof) Let $I(P) = \{Q \in Po(E) : Q \leq P\}$ be a transitive convex geometry, and $G(P)$ be the corresponding transitively closed acyclic digraph of $P$. Let $N = \{e_1 < e_2 < \cdots < e_n\}$ be a total order on $E$ which is compatible with $P$. Then the convex geometry $I(P)$ is a deletion of the convex geometry $I(N)$. Hence $I(P)$ is itself a convex antimatroid.

(6) Lattices of closure operators on posets:
The set of closure operators on a poset $P$ is a join-distributive lattice when partially ordered by $f \leq g \iff f(x) \subseteq g(x)$ ($\forall x \in P$), and furthermore forms an antimatroid (Stern [152], p.163). It is known to be a supersolvable lattice (Hawrylysz and Reiner [89]).

(7) $k$-family of a poset:
Let $P$ be a poset. For a couple of anti-chains $A, B$ in $P$, we set a partial order $A \preceq B$ if for any $a \in A$, there exists $b \in B$ such that $a \leq b$.

Let $A_k(P)$ denote the set of subsets of $P$ in which any chain is of height at most $k$. Note that $A_1(P)$ is the collection of anti-chains of $P$. Each element $X$ of $A_k(P)$ has a unique canonical partition
\{X_1, \ldots, X_k\} \text{ into anti-chains where a partition is canonical if } i \leq j, \text{ then for any } a \in X_i, \text{ there exists } b \in X_j \text{ such that } a \preceq_P b.

Let \{A_1, \ldots, A_k\} and \{B_1, \ldots, B_k\} be the canonical partition of \(k\)-families \(A\) and \(B\), respectively. We shall define \(A \preceq B\) if for any \(i = 1, \ldots, k\), \(A_i \preceq B_i\). Then \(\preceq\) is a partial order on \(A_k(P)\), and \(A_k(P)\) is a join-distributive lattice with respect to this partial order [84]. Even more \(A_k(P)\) forms an antimatroid by Proposition 2.27 of Greene and Kleitman [84].

(8) Convex sets in a join-semilattice:
A subset \(S\) of a join-semilattice \(L\) is convex if \(y \in L\) is in \(S\) provided that \(x \leq y \leq z\) for some \(x, z \in S\). Then the set of convex sets of a join-semilattice constitutes a convex geometry if and only if the semilattice is a tree-diagram lattice, i.e. a rooted tree (Libkin and Gurvich [117]).

A shelling giving rise to a corresponding antimatroid is the deletion of meet-irreducible elements.

8.5 Chordal graphs

(1) Chordal convex geometries: (See p.66.)
For a chordal graph \(G\), let us define precisely once again a chordal convex geometry \(\mathcal{K}_G \subseteq 2^{V(G)}\) and a chordal antimatroid \(\mathcal{F}_G \subseteq 2^{V(G)}\) as follows.

\[
\mathcal{F}_G = \{\{a_1, \ldots, a_k\} : a_1 \cdots a_k \text{ is the prefix of a simplicial shelling sequence of } G\}\]
\[
\mathcal{K}_G = \{U \subseteq V(G) : U \text{ is a monophonically convex set of } G\}\]

Then

**Theorem 8.7 (Faber and Jamison [73])** Let \(G\) be a chordal graph. Then \(\mathcal{K}_G\) is a convex geometry on \(V(G)\), \(\mathcal{F}_G\) is an antimatroid on \(V(G)\), and they are the dual to each other.

We can restate Theorem 3.6 in terms of monophonically convex sets of chordal graphs.

**Proposition 8.8** Let \(G\) be a chordal graph, and \(\mathcal{K}\) the chordal convex geometry of \(G\). For \(u, v \in V(G)\), let \([u, v]_G\) denote the set of vertices on the chordless paths connecting two vertices \(u\) and \(v\). Suppose \(X \subseteq V(G)\) is a monophonically convex set of \(G\). Then \(x \in \text{ex}(X)\) if and only if \(x\) is a simplicial vertex of the induced subgraph \(G_X\). For every \(a \in X\), \(a \not\in \text{ex}(X)\), there exist \(p, q \in \text{ex}(X)\)
such that $a \in [p, q]_G$. Moreover $(\{p, q\}, a)$ is a rooted circuit of the chordal convex geometry of $G$, and every rooted circuit arises in this way.

(Proof) As is described in the proof of Theorem 3.6, the deletion of a simplicial vertex of a chordal graph result in a chordal graph again. Hence the first assertion follows.

The last assertion is obvious from Theorem 3.6. □

(2) Strongly chordal graph:
For two vertices $u, v$ in a graph $G$, $d_G(u, v)$ denotes the distance between $u$ and $v$ in $G$. Let $C$ be a cycle of even length in $G$. Then an edge of $G$ is a odd chord of $C$ if it connects two vertices $u, v$ in $C$ and $d_C(u, v)$ is odd with $d_C(u, v) \geq 3$.

A graph is a strongly chordal graph if it is chordal and every even cycle of length at least six has a odd chord. A vertex $v$ of a graph is simple provided that for any vertices $x, y$ in the neighborhood $N(v)$, either $N(x) \subseteq N(y)$ or $N(y) \subseteq N(x)$ holds, i.e. $\{N(w) : w \in N(v)\}$ is linearly ordered with respect to inclusion relation. For more details, see p.78.

(3) Ptolemaic convex geometries:
A subset $X$ of vertices of a graph $G$ is geodecically convex if every middle-point on a shortest path connecting two vertices in $X$ lies in $X$.

For a graph $G$ and two vertices $x, y \in V(G)$, $d(x, y)$ is the length of shortest paths between $x$ and $y$. In case that $x$ and $y$ are not in the same connected components, we define $d(x, y) = +\infty$.

A graph $G$ is said to be a Ptolemaic graph if for every four vertices $x, y, z, w$, it holds that

$$d(x, y)d(z, w) \leq d(x, z)d(y, w) + d(x, w)d(y, z).$$

Ptolemaic graphs form a subclass of chordal graphs.

**Theorem 8.9 (Farber and Jamison [73])** For a graph $G$, the following are equivalent.

(i) $G$ is a Ptolemaic graph.

(ii) The geodecically convex sets of $G$ constitutes a convex geometry.

(iii) $G$ is a chordal graph, and every cycle of length 5 has at least three chords.

(iv) $G$ is a chordal graph, and does not contain 3-fan (Fig. 8.16) as an induced subgraph.

(v) $G$ is a chordal graph, and every chordless path is a shortest path.

(vi) $G$ is a chordal graph, and the collection of monophonically convex sets and that of geodecically convex sets are equal.

Let us call the convex geometry defined in the above theorem a Ptolemaic convex geometry. It follows from Theorem 8.9 that a Ptolemaic convex geometry is a special case of chordal convex geometries.

**Proposition 8.10** A Ptolemaic graph is a strongly chordal graph.

(Proof) Proof!, or 参照。Ptolemaic graph が長さ 6 のサイクルを含むとき、strong chord が存在視することを示し、そこから induction を使う。 □
(4) Block convex geometries: A block graph is a graph such that every maximal 2-connected component is a complete graph.

**Theorem 8.11 (Jamison [96])** The collection of vertices of connected induced subgraph of a graph $G$ is a convex geometry if and only if $G$ is a block graph.

A block graph is obviously a chordal graph, and the simplicial shelling gives an antimatroid dual to the convex geometry of the block graph.

(5) Node-shelling of a tree (Boulaye 1967, 1968): The node-shelling antimatroid of a tree is a simple special case of chordal antimatroids. The shelling of deleting extreme vertex of a tree gives rise to an antimatroid. The corresponding convex geometry is composed of the set of vertices of subtrees of a tree. Since a tree is a special case of chordal graphs, it is clear that a node-shelling antimatroid of a tree is a chordal antimatroid.

In a node-shelling convex geometry of a tree, rooted circuits and rooted cocircuits are easily described. For a tree $T = (V, E)$ and any vertex $x \in V$, let $C(x)$ and $D(x)$ denote the stem clutter and the costem clutter of an element $x$. If $x$ is an extreme vertex, then $C(x) = \emptyset$ and $D(x) = \{\emptyset\}$. Otherwise, deleting $x$ from $T$ gives a partition into a couple of connected subtree. Let $V_1, V_2$ be the vertex sets of these subtrees. Then

$$C(x) = \{\{y, z\} : x \text{ is a middle point of a path connecting } y, z,\},$$

$$D(x) = \{V_1, V_2\}$$

A critical rooted circuit of the node-shelling of a tree $T$ is a path of length 2. That is, if $\{x, y\}, \{y, z\} \in E(T)$ and $x, y, z$ are all distinct, then $(\{x, z\}, y)$ is a critical rooted circuit.

The convex geometry dual to a node-shelling antimatroid of a tree a node-shelling is said to be a node-shelling convex geometry of a tree.

Fig. 8.17 shows a tree $K_{1,3}$, and the repetition of deletion of an extreme vertex of $K_{1,3}$ will give rise to a node-shelling antimatroid in the right-hand side of Fig. 8.17.

(6) Edge-shelling of a tree: In contrast to a node-shelling, we can similarly define an edge-shelling antimatroid of a tree and its dual convex geometry. This can be naturally extended to the case of forests, and so an edge-shelling of a forest gives an antimatroid on the set of edges.

As is easily checked, the line graph of a tree is a connected block graph. For example, see Fig. 8.18 and 8.19. As is stated in Section 3.1, a line graph is claw-free. Furthermore,
**Proposition 8.12 ([88])** A graph is the line graph of a tree if and only if it is a claw-free connected block graph.

**Corollary 8.13** An antimatroid is an edge-shelling antimatroid of a tree if and only if it is a simplicial shelling antimatroid of a connected claw-free block graph.

Hence the class of edge-shelling convex geometries of trees is a subclass of that of block graph convex geometries.

The classes of chordal graphs and other graphs are in the following subclass relations.

\[
\text{chordal} \supset \text{strongly chordal} \supset \text{Ptolemaic} \supset \text{block}
\]

### 8.6 Miscellaneous:

1. \textit{k-degree shelling of a graph} (Ardila and Maneva [8]):
   Given a graph \( G = (V, E) \), deleting a vertex of degree at most \( k \) is a shelling process, and it gives rise to an antimatroid on \( V \). This shelling eliminates all the vertices if and only if \( G \) contains no induced subgraph in which the degree of each vertex is at least \( k + 1 \). In a special case that \( G \) is a tree and \( k = 1 \), this shelling process is the same with the node-shelling of a tree.
8.7 Simplicial shelling and simple shelling

(1) Simplicial vertex shelling and simple vertex shelling of graphs:
A vertex of a graph \( G = (V,E) \) is a simplicial vertex if the neighbourhood is a clique. The sequential elimination of simplicial vertices is a shelling process. Hence it gives rise to an antimatroid, which we call a simplicial antimatroid of \( G \).

Let \( N(v) = \{v\} \cup \{w \in V : vw \in E\} \) be the neighbourhood of a vertex \( v \). If there is a suitable indexing \( N(v) = \{v\} \cup \{w_1, \ldots, w_p\} \) such that \( N(w_1) \subseteq N(w_2) \subseteq \cdots \subseteq N(w_p) \), \( v \) is called a simple vertex of \( G \). Similarly, the sequential elimination of simple vertices is a shelling process, and so establishes an antimatroid, which we call a simple antimatroid of \( G \). Let \( A_S(G) \) and \( A_P(G) \) be a simplicial antimatroid and a simple antimatroid of \( G \), respectively. Since a simple vertex is necessarily a simplicial vertex, a simple vertex elimination is a simplicial vertex elimination. Hence \( A_P(G) \subseteq A_S(G) \) holds. The bull graph of Fig. 8.20 is a typical example of a chordal graph for which \( A_P(G) \neq A_S(G) \). The simplicial vertex shelling and the simple vertex shelling of the bull graph are distinct as the vertex \( v \) is simplicial but not simple.

![Figure 8.20: The bull graph](image1)

![Figure 8.21: A bull-convex free graph](image2)

(2) Convex-bull-free chordal graphs:
Let \( G = (V,E) \) be a chordal graph. As is stated above, \( A_P(G) \subseteq A_S(G) \) holds. We shall consider the condition that \( A_P(G) = A_S(G) \).

**Theorem 8.14** Let \( G \) be a chordal graph. Then \( A_P(G) = A_S(G) \) if and only if \( G \) includes no monophonically convex set of vertices whose induced subgraph is isomorphic to the bull graph with the vertex ‘\( v \)’ being simple.

**Proof** If \( A_P(G) \neq A_S(G) \), there exist a simplicial vertex shelling sequence which is not a simple vertex shelling. Hence there exists such a simplicial shelling sequence \( \alpha = u_1u_2\cdots u_n \) with \( [\alpha] = V \) that \( u_k \) is simplicial but not simple for some \( 1 \leq k \leq n \). Put \( v = u_k \), and \( N(v) = \{u\} \cup \{w_1, \ldots, w_m\} \). Obviously \( m \geq 2 \) as \( v \) is not simple. Since \( v \) is not a simple vertex, there exist distinct \( w_i, w_j \in N(v) \) such that \( N(w_i) \nsubseteq N(w_j) \) and \( N(w_j) \nsubseteq N(w_i) \). Hence we can take \( x_i \in N(w_i) \setminus N(w_j) \) and \( x_j \in N(w_j) \setminus N(w_i) \). By definition, the edges \( x_iw_j \) and \( x_jw_i \) do not exist in \( G \). If the edge \( x_ix_j \) exists, there will be a cycle \( (x_i, w_i, x_j, w_j) \) without chord, which contradicts the chordality of \( G \). Hence the edge \( x_ix_j \) does not exist. Furthermore, if either the edge \( vx_i \) or \( vx_j \) exists, \( v \) is no more a simplicial vertex, a contradiction. Thus the induced graph \( G[[v,w_i,w_j,x_i,x_j]] \) is isomorphic to the bull graph. By definition \( \{u_k, u_{k+1}, \ldots, u_n\} \) is a monophonically convex set in \( G \).

Conversely, if \( A_P(G) = A_S(G) \), every simplicial shelling is a simple shelling. Hence there is no monophonically convex set which induces the bull graph. Otherwise, there is a simplicial vertex shelling which is not a simple vertex shelling, which contradicts the assumption. \( \square \)
Let us call a chordal graph $G$ a convex-bull-free chordal graph. A bull-free chordal graph is a chordal graph which has no induced graph isomorphic to the bull graph. Obviously,

**Corollary 8.15** A bull-free chordal graph is a convex-bull-free chordal graph.

But the converse is not true. Actually the graph of Fig. 8.21 is a convex-bull-free chordal graph, but not a bull-free chordal graph.

### 8.8 Totally balanced hypergraphs and strongly chordal graphs

(1) Totally balanced hypergraphs and nest vertex shelling ([73]):

Let $\mathcal{H} = (V, \mathcal{E})$ be a totally balanced hypergraph. A vertex $x \in V$ is a nest vertex if $\mathcal{E}_x = \{ H \in \mathcal{E} : x \in H \}$ is a nested family. That is, we can take indexes so that $\mathcal{E}_x = \{ F_1, F_2, \ldots, F_m \}$ and $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_m$.

A path in a hypergraph $\mathcal{H} = (X, \mathcal{E})$ is a sequence $x_1E_1x_2E_2 \cdots x_{n-1}E_nx_n$ such that $x_i, x_{i+1} \in E_i$ for $i = 1, \ldots, n-1$, $x_i$’s are pairwise distinct and $E_i$’s are pairwise distinct. A circuit is similarly defined except $x_1 = x_n$. The circuit(path) is simple if $E_i \cap \{ x_1, x_2, \ldots, x_n \} = \{ x_i, x_{i+1} \}$ for $i = 1, 2, \ldots, n-1$.

A hypergraph $\mathcal{H} = (X, \mathcal{E})$ is totally balanced if it contains no simple circuit of length at least 3.

**Theorem 8.16** ([73]) A hypergraph $\mathcal{H}$ is totally balanced if and only if every subhypergraphs of $\mathcal{H}$ has a nest vertex.

Hence we can consider the nest vertex shelling of a totally balanced hypergraph, so that it gives an antimatroid. We shall call it a nest antimatroid of $\mathcal{H}$.

**Theorem 8.17** (Faber and Jamison [73]) In a totally balanced hyper graph, every nonnest vertex lies on a simple path between nest vertices.

Hence a nest antimatroid has the path-property.

We say that a vertex set $X \subseteq V$ is simple-path convex if the vertices on a simple path connecting two vertices in $X$ are all contained in $X$.

**Corollary 8.18** ([73]) The collection of the simple-path convex sets of a hypergraph $\mathcal{H}$ is a convex geometry if and only if $\mathcal{H}$ is totally balanced.

**Corollary 8.19** ([73]) For a totally balanced hypergraph $\mathcal{H}$, the convex geometry composed of the simple-path convex sets is dual to the nest antimatroid.

(2) Strongly chordal convex geometries:

For the definition, see p.74.

**Theorem 8.20** (Faber [72], Faber and Jamison [73]) Then the following conditions are equivalent for a graph $G = (V, E)$.
(i) $G$ is a strongly chordal graph.
(ii) Every induced subgraph of $G$ has a simple vertex.
(iii) The neighborhood hypergraph $\mathcal{H} = (V, \mathcal{N}(G))$ is totally balanced where $\mathcal{N}(G) = \{N(v) : v \in E\}$.

Hence the simple element shelling process eliminates all the vertices of a strongly chordal graph.

A chordal graph is said to be bull-free if it includes no induced subgraph isomorphic to the bull graph.

**Theorem 8.21** A bull-free chordal graph is a strongly chordal graph.

(Proof) Let $G$ be a bull-free chordal graph, and suppose that it is not strongly chordal. Then there is an even cycle $(v_1, v_2, \ldots, v_k, v_{k+1}(= v_1))$ with $k$ at least six such that there is no odd chord. Since $G$ is chordal, there must exist a triangle $(v_{i-1}, v_i, v_{i+1})$ with $v_{i-1}v_{i+1} \in E(G)$. (Indexes are counted by modulo $k$.) Since there is no odd chord, neither the chord $v_{i-2}v_{i+1}$ nor $v_{i-1}v_{i+2}$ exists. If there is a chord $v_{i-2}v_{i+1}$, there will be a cycle $(v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2})$. Then there should be the chord either $v_{i-2}v_{i+1}$ or $v_{i-1}v_{i+2}$ as $G$ is chordal, which is a contradiction. Hence there is no chord between $v_{i-2}$ and $v_{i+2}$. Then the induced graph $G[\{v_{i-2}, v_{i-1}, v_{i}, v_{i+1}, v_{i+2}\}]$ is isomorphic to the bull graph. It is a contradiction, and this completes the proof. \(\square\)

**Corollary 8.22** The simplicial shelling antimatroids of bull-free chordal graph constitute the subclass of the simple vertex shelling antimatroids of strongly chordal graphs.

We note that a simple vertex $v$ of a strongly chordal graph $G$ is a nest vertex of the totally balanced hypergraph $\mathcal{N}(G)$.

A path $P = u_0u_1\ldots u_n$ in a graph is even-chorded path if it has no chord of odd length and the end-vertices $u_0, u_n$ are not adjacent to any chord.

A vertex set $K \subseteq V$ is strongly convex if every vertex lying on an even-chorded path connecting vertices in $K$ necessarily belongs to $K$.

**Theorem 8.23** ([73]) For a strongly convex graph, every nonsimple vertex lies on an even-chorded path between simple vertices.

**Corollary 8.24** ([73]) The strongly convex sets of a graph $G$ constitute a convex geometry if and only if $G$ is a strongly chordal graph.

We shall call it a strongly chordal convex geometry. The antimatroid dual to it is called a strongly chordal antimatroid. By Theorem 8.23, the shelling determining a strongly chordal antimatroid is the deletion of simple vertices. Since a simple vertex is simplicial, the shelling process of deleting simple vertices and deleting simplicial vertices give rise to different antimatroids generally. In other words, the strongly convex geometry of a strongly chordal graph $G$ is a sublattice of the chordal convex geometry of $G$.

The graph of Fig. 8.22 is a strongly chordal graph, and $a, b$ are simple vertices. $c$ is a simplicial vertex, but not simple. We can check that the strongly chordal convex geometry of Fig. 8.22 is not
isomorphic to any chordal convex geometry of a chordal graph. In contrast, the graph of Fig. 8.23 is chordal, but not strongly chordal. The chordal convex geometry of Fig. 8.23 is not isomorphic to any strongly chordal convex geometry. Hence the classes of chordal convex geometries and strongly chordal convex geometries do not include each other as a subclass.

![Figure 8.22: A strongly chordal graph](image1)

![Figure 8.23: A chordal graph](image2)

The inclusion relation of classes of convex geometries arising from simplicial shelling and simple shelling of graphs is indicated in Fig. 8.24.

### 8.9 Characteristic paths and shelling vertices

First we list up the convex geometries and antimatroids satisfying the path-property. By Corollary 7.27, they are automatically atomistic.

**Example 8.25**

- Node-shelling and edge-shelling of trees.
- Chordal convex geometries of chordal graphs.
- Double shelling of posets.
- Transitivity antimatroids of transitively closed digraphs.
- Simple-path convex geometries of totally balanced hypergraphs.
- Simple vertex shelling antimatroid of strongly chordal graphs.

It could come up to our mind that we have repeated similar arguments for chordal convex geometries and others. There is some specific property (P) of the paths in a graph. Let us call such a path a P-path that satisfies the property (P). For a graph $G$, a subset $K$ of the vertex set $V(G)$ is defined to be convex (or closed) if every middle point on a P-path between two vertices in $K$ also belongs to $K$. In case of chordal convex geometries, a P-path is a chordless path. Similarly, the property (P) for Ptolemaic convex geometries is that the path should be the shortest path. For strongly chordal convex geometries, a P-path is an even-chorded path. We describe in the Table 8.1 the characteristic property (P) on paths and the removable vertex in shelling processes.
Figure 8.24: Relations of classes of convex geometries

Table 8.1: Table of objects and characteristic concepts

<table>
<thead>
<tr>
<th>Object</th>
<th>Convex set</th>
<th>Characteristic path</th>
<th>Shelling</th>
</tr>
</thead>
<tbody>
<tr>
<td>chordal graph</td>
<td>monophonically convex</td>
<td>chordless path</td>
<td>simplicial vertex</td>
</tr>
<tr>
<td>strongly chordal graph</td>
<td>strongly convex</td>
<td>even-chorded path</td>
<td>simple vertex</td>
</tr>
<tr>
<td>Ptolemaic graph</td>
<td>geodecically convex</td>
<td>shortest path</td>
<td>simplicial vertex</td>
</tr>
<tr>
<td>block graph</td>
<td>connected induced</td>
<td></td>
<td>simplicial vertex</td>
</tr>
<tr>
<td></td>
<td>subgraph</td>
<td></td>
<td></td>
</tr>
<tr>
<td>totally balanced</td>
<td>simple-path convex</td>
<td>simple-path</td>
<td>nest vertex</td>
</tr>
<tr>
<td>hypergraph</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
8.10 Searches on rooted graphs

(1) Node-search of digraphs and graphs [112]:

For a rooted digraph \( G = (V \cup \{r\}, A) \), search sequences starting from the root give rise to an antimatroid on the vertex set \( V \). More precisely, let us define

\[
\delta(X) = \{ v \in V \setminus X \mid (u, v) \in A \text{ for some } u \in (V \setminus X) \cup \{r\} \} \quad \text{for } X \subseteq V.
\] (8.3)

For simplicity, we assume that every vertex is reachable from the root \( r \) through a directed path. Then we can define a shelling process on \( V \).

\[
\begin{align*}
\alpha &:= r; \\
X &:= \{r\} \\
\text{while}(\delta(X) \neq \emptyset) &\text{ do begin} \\
&\quad \text{Choose arbitrarily a vertex } x \text{ in } \delta(X); \\
&\quad \alpha := \alpha x; \\
&\text{end;}
\end{align*}
\]

The collection of the sequences of these shelling sequences neglecting the root \( r \) gives an antimatroid. For instance, this shelling of the rooted digraph \( G_3 \) of Fig. 8.25 defines a node-search antimatroid of the right-hand side of Fig. 8.25.

In an undirected rooted graph, everything is similar and an antimatroid can be defined on the vertex set. We only have to replace each undirected edge \( \{a, b\} \) with a pair of directed edges \((a, b)\) and \((b, a)\).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{G3}
\caption{A rooted digraph \( G_3 \) and its node-search antimatroid}
\end{figure}

(2) Edge-search of rooted digraphs and rooted graphs [112]:

Just replacing nodes with edges, an edge-search could be easily defined, which gives rise to antimatroids on the edge sets of digraphs and undirected graphs both.

8.11 Characterizations of search antimatroids

In a finite lattice, an element \( z \neq 0 \) is called a cycle if the interval \([0, z]\) is a chain. If \( z \) is a cycle, it is automatically a join-irreducible element

**Theorem 8.26 ([130])** Let \( L \) be a join-distributive finite lattice. Then the following are equivalent.
(1) \( L \) is isomorphic to an antimatroid derived from node-search on a rooted digraph,

(2) \( L \) does not contain \( D_5 \) (Fig.8.26) as an interval minor,

(3) \( L \) Every element \( x \ (x \neq 0) \) of \( L \) is a join of cycles.

**Theorem 8.27 ([130])** Let \( L \) be a join-distributive finite lattice. Then the following are equivalent.

(1) \( L \) is isomorphic to an antimatroid arising from node-search on a rooted undirected graph.

(2) \( L \) does not contain \( D_5 \) or \( S_{10} \) (Fig.8.27) as an interval minor.

\[ \begin{align*}
\{x,y,z\} & \quad \{x,y,z\} \\
\{x,z\} \quad \{x,y\} \quad \{y,z\} & \quad \{x,y\} \\
\{x\} \quad \emptyset & \quad \emptyset \\
\end{align*} \]

**Figure 8.26: \( S_7 \) and \( D_5 \)**

\[ \begin{align*}
{a,c,d} & \quad \{a,b,c,d\} \\
{a,c} \quad \{a,b\} \quad \{a,b,d\} & \quad \{a,b\} \\
\{a\} \quad \emptyset & \quad \emptyset \\
\end{align*} \]

**Figure 8.27: \( G_4 \) and \( S_{10} \)**

**Theorem 8.28 ([130])** Let \( L \) be a join-distributive finite lattice. Then the following are equivalent.

(1) \( L \) is a distributive lattice,

(2) \( L \) does not contain \( S_7 \) (Fig.8.26) as an interval minor.

Based on Theorem 8.26, we can induce a forbidden-minor characterization of an line-search antimatroid of a rooted digraph. Actually let \( G \) be a rooted line graph of a rooted multi-digraph \( H \). Then the line-search antimatroid of \( H \) is the node-search antimatroid of \( G \).

**Proposition 8.29 ([90, 139])** Let \( G = (V, E) \) be a digraph. Then \( G \) is a line graph if and only if for every \( x, y, z, w \in V \), \( xy, zy, zw \in E \) implies \( xw \in E \).
A line-graph can be characterized by the above condition, which we call a *Heuchenne condition*.

In a digraph, we call an edge a redundant edge if there is no directed path without short-cut containing e. A digraph is irredundant if it contains no redundant edge. Then there is a one-to-one correspondence between rooted multi-graphs and irredundant rooted digraphs satisfying Heuchenne condition. Hence in order to characterize line-search antimatroids of rooted digraphs, it is sufficient to characterize node-search antimatroids of irredundant rooted digraphs violating Heuchenne condition.

Let $G = (V, E)$ be a rooted digraph with root $r \in V$, and $A(G)$ be the node-search antimatroid of $G$. For $A, B \in A(G)$ with $A \subseteq B$, delete the vertices in $V - B$ together with the edges adjacent to them, and contract the vertices in $A$ to a root $r$ as well as deleting redundant edges. Then the resultant rooted digraph is called a rooted minor and denoted $G[A, B]$.

Before stating the main theorem, we define four types of rooted digraphs $G = (V, E, r)$.

1. **Type $A$**: $G_A = (V_A, E_A, r)$ with $V_A = \{r, a, b, c, d\}$ and $E_A = \{ra, rb, ac, bc, bd\}$.
   $G_A$ is described in Fig. 8.28. A digraph of type $A$ violates the Heuchenne condition.

2. **Type $B$**: $G_B = (V_B, E_B, r)$ with $V_B = \{r, a, b, c, d\}$ and $E_B = \{ra, rb, ac, bc, bd, dc\}$.
   $G_B$ is described in Fig. 8.29. A digraph of type $B$ violates the Heuchenne condition.

3. **Type $C_{m,n}$**: $G_C = (V_C, E_C, r)$ ($m, n \geq 1$) with
   
   $V_C = \{r, a, b, c(= x_0), d(= y_0), e, x_1, \ldots, x_{m-1}, y_1, \ldots, y_{n-1}\}$,
   
   $E_C = \{ra, rb, ac, bd, ce, ed, cx_1, x_1x_2, \ldots, x_{m-2}x_{m-1}, x_{m-1}e, dy_1, y_1y_2, \ldots, y_{n-2}y_{n-1}, y_{n-1}e\}$.

4. **Type $D_{l,m,n}$**: $G_D = (V_D, E_D, r)$ ($l, m, n \geq 1$) with
   
   $V_D = \{r, a, b, c(= x_0), d(= y_0), e, f(= z_0)\}$
   
   $E_D = \{ra, rb, ac, bd, ec, ed, dy_1, y_1y_2, \ldots, y_{m-2}y_{m-1}, y_{m-1}e, cz_1, x_1x_2, \ldots, x_{l-2}x_{l-1}, x_{l-1}f\}$
   
   $y_1y_2, \ldots, y_{m-2}y_{m-1}, y_{m-1}f, z_1z_2, \ldots, z_{n-2}z_{n-1}, z_{n-1}e\}$.

**Theorem 8.30** ([134]) Let $G$ be an irredundant rooted digraph. Then $G$ is a rooted line graph if and only if $G$ contains no rooted minor isomorphic to $A$, $B$, $C_{m,n}$, $D_{l,m,n}$ ($l, m, n \geq 1$).

**Corollary 8.31** ([134]) An antimatroid is a line-search antimatroid of a rooted digraph if and only if it contains no interval minor isomorphic to $D_5$ nor isomorphic to a node-search antimatroid of $A$, $B$, $C_{m,n}$, $D_{l,m,n}$ ($l, m, n \geq 1$).
Chapter 9

Representation Theorem

9.1 Lifting of antimatroids

We shall investigate a single-element extension of antimatroids. Let $\mathcal{A}_1$, $\mathcal{A}_2$ be the subfamilies of an antimatroid $\mathcal{A}$, and suppose that they satisfy

(E0) $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}$,
(E1) $\mathcal{A}_1$ is an antimatroid,
(E2) $\mathcal{A}_2$ is a filter in $\mathcal{A}$,
(E3) $\mathcal{A}_2 = \{ Y \in \mathcal{A} : X \subseteq Y \text{ for some } X \in \mathcal{A}_1 \cap \mathcal{A}_2 \}$.

Let $p$ be a new element not in $E$. Then we can define a one-rank higher lattice by

$$(A \uparrow p)_{(\mathcal{A}_1, \mathcal{A}_2)} = \mathcal{A}_1 \cup (\mathcal{A}_2 + p) = \mathcal{A}_1 \cup \{ Y \cup p : Y \in \mathcal{A}_2 \} \quad (9.1)$$

Let us call (9.1) a lifting of $\mathcal{A}$ at $(\mathcal{A}_1, \mathcal{A}_2)$ by adding $p$. We write $\mathcal{A} \uparrow p$ to denote $(A \uparrow p)_{(\mathcal{A}_1, \mathcal{A}_2)}$ when no confusion may occur.

**Theorem 9.1** A lifting $(A \uparrow p)_{(\mathcal{A}_1, \mathcal{A}_2)}$ is an antimatroid on $E \cup p$.

(Proof) $\emptyset \in \mathcal{A} \uparrow p$ is obvious. To see (L2), take any $X \in \mathcal{A} \uparrow p$. If $X \in \mathcal{A}_1$, (L2) is clear. Otherwise suppose $X = X' \cup p$ and $X' \in \mathcal{A}$. If $X'$ is not minimal in $\mathcal{A}_2$, there exists an element $x \in X'$ such that $X' \setminus x \in \mathcal{A}_2$, and we have $X \setminus x = (X' \setminus x) \cup p \in \mathcal{A} \uparrow p$. If $X'$ is minimal in $\mathcal{A}_2$, $X' \in \mathcal{A}_1$ follows from (E3). Hence $X \setminus p = X' \in \mathcal{A}_1 \subseteq \mathcal{A} \uparrow p$. So (L2) holds. Finally we shall show (L3'). The only interesting case is that $X \in \mathcal{A}_1$ and $Y = Y' + p \in \mathcal{A}_2 + p$. By (E2) $\mathcal{A}_2$ is a filter, and we have $X \cup Y' \in \mathcal{A}_2$. Hence $X \cup Y = (X \cup Y') + p \in \mathcal{A} \uparrow p$. \hfill \Box

A trace minor by one element and a lifting are the reverse operation of each other. Take an element $p \in E$. Then we have a one-rank lower lattice

$$\mathcal{A} - p = \{ X \setminus p : X \in \mathcal{A} \}. \quad (9.2)$$

$\mathcal{A} - p$ is an antimatroid on $E \setminus p$, and we call it a reduction of $\mathcal{A}$ at $p$. A reduction and a lifting are converses.
Theorem 9.2 ([128]) The following hold.

(a) For any $p \in E$, we have

$$((A - p) \uparrow p)_{(A_1, A_2)} = A.$$  \hspace{1cm} (9.3)

where $A_1 = \{X : X \in A, p \notin X\}$, $A_2 = \{X - p : X \in A, p \in X\}$.

(b) Conversely, take a new element $q$ not in $E$, and suppose $A_1$ and $A_2$ satisfies (E0), (E1), (E2) and (E3). Then

$$((A \uparrow q)_{(A_1, A_2)}) - q = A.$$  \hspace{1cm} (9.4)

From Theorems 9.1 and 9.2, we have

Corollary 9.3 Let $A$ be a family of subsets of $E$. Then $A$ is an antimatroid if and only if it can be constructed from a chain by applying lifting repeatedly.

(Proof) Order arbitrarily the elements of $E$ as $p_1, p_2, \ldots, p_n$. Then $(\cdots ((A - p_1) - p_2) \cdots) - p_n$ is a trivial lattice, and repeating the reverse lifting $n$ times gives $A$. \hspace{1cm} \boxdot$

Let $A$ be an antimatroid on $E$, and $A$ a feasible set of $A$. When we define $A_1$ and $A_2$ by

$$A_1 = A, \quad A_2 = [A, E],$$ \hspace{1cm} (9.5)

then (E0), (E1), (E2) and (E3) are trivially satisfied, and we shall call the resultant lifting a 1-lifting. Further more if $E \setminus A \in A$ holds here, we call it a self-dual 1-lifting.

Theorem 9.4 ([128]) $A \subseteq 2^E$ is a poset antimatroid if and only if it is obtained from a trivial lattice $\{\emptyset\}$ by repeating 1-lifting.

A rooted tree is a tree with a specified vertex called a root. A rooted-tree poset is a poset such that the Hasse diagram is a tree having the root as the maximum element.

Corollary 9.5 ([128]) $A$ is a poset antimatroid of a rooted-tree poset if and only if it is obtained from a trivial lattice $\{\emptyset\}$ by repeating self-dual 1-lifting.

We further define another lifting. Suppose $A$ and $E \setminus A$ are both nonempty feasible sets of $A$. Then the families

$$A_1 = A, \quad A_2 = [A, E] \cup [E \setminus A, E]$$ \hspace{1cm} (9.6)

satisfy (E0), (E1), (E2) and (E3). This determines another single element extension of $A$, which we shall call a 2-lifting.

Then the line-shelling antimatroid of a tree can be characterized by 2-lifting as follows.

Theorem 9.6 ([128]) Suppose $k \geq 0$, $m \geq 1$. $A$ is an antimatroid of edge shelling of a tree of $m$ end-edges and $k$ interior edges if and only if $A$ can be constructed from a Boolean algebra $2^{[m]}$ by applying 2-lifting $k$ times.
9.2 Representation theorem of convex geometries

Def. 9.7 \((E, R)\) を \(\mathbb{R}^n\) 中の有限集合で互いに素なものとし、かつ常に \(R \neq \emptyset\) とする。このペア \((E, R)\) を点のペア配置もしくは単に点配置と呼ぶことにする。以下簡単のために \(E \cap \text{conv}(R) = \emptyset\)であるとしておく。

以下 \((E, R)\) から定まる \(E\) 上の affine kernel–shelling の凸幾何とアンチマトロイドを考える。

Def. 9.8 \((X, e)\) が根付き集合のとき \(e\) を根、\(X\) を茎 stem と呼ぶことにする。

Def. 9.9 あるアンチマトロイドのある元 \(e\) を根とするサーキットの stem の全体のなすクラックを stem clutter と呼ぶ。

Def. 9.10 有限集合 \(S \subseteq \mathbb{R}^n\) と 1 点 \(v \in \mathbb{R}^n\) に対して、\(v \not\in \text{conv}(S)\) であるとき、\(v\) を原点とする錐 cone \(C^+(v, S), C^-(v, S)\) を以下で定義する。

\[
C^+(v, S) = \{ (1 - \lambda)v + \lambda a \mid \lambda \geq 0, a \in \text{conv}(S) \}
\]
\[
C^-(v, S) = \{ (1 - \lambda)v + \lambda a \mid \lambda \leq 0, a \in \text{conv}(S) \}
\]

Lemma 9.11 \(v \in E, A \subseteq E \setminus v\) に対して、以下は同値になる。

(1) \((A, v)\) が kernel–shelling on \((E, R)\) の凸幾何のサーキットになる。

(2) \(A\) は次を満たすかこの性質に関して極小である。

\[
\text{conv}(A) \cap C^-(v, R) \neq \emptyset
\]

(Proof) \(v \in \text{conv}(A \cup R) \iff \text{conv}(A) \cap C^-(v, R)\) は自明に成立する。これとサーキットの定義から補題は明らか。

9.3 Proof of the representation theorem

\((K, E)\) を loop-free な凸幾何とする。

ここでセルフループ \(e \in E\) が存在すると \((\emptyset, e) \in C\) となり、\(r = r(\emptyset, e) = e\) となって \(e\) が核 \(R\) に含まれることになります。補題 9.12 で言えば、

\[
\text{conv}(R) \cap E = \text{self-loop の全体 } \ni e
\]

となり、共通部分が空ではなくなるので、最初から loop-free と仮定しておきます。

\(E\) の要素を \(|E| - 1\) 次元のアフィン空間の \(E\) 次元単体の頂点集合と同一視する。

各根付きサーキット \((X, e) \in C(F)\) に対して、空間中の点

\[
r(X, e) = (|X| + 1)e - \sum_{x \in X} x
\]

に対応させる。このとき \(R = \{r(X, e) : (X, e) \in C(F)\}\) を核とした \(E\) 上の凸シェリングアンチマトロイドは、もとの \(F\) に一致する、ことを示す。
Lemma 9.12 ([102])

conv(R) ∩ E = ∅

（Proof）あるv ∈ Eに対してv ∈ conv(R)であったとする。すると、ある部分集合R' ⊆ Rがあってvがすべての頂点集合でかつること、つまり

\[ v = \sum_{e \in R'} k_e a = \sum_{e \in R'} k_e \left( (|X_e| + 1)e - \sum_{x \in X_e} x \right) \]

となりこれを書き直すと、

\[ v + \sum_{e \in R'} k_e \left( \sum_{x : x \in X_e} \right) = \sum_{e \in R'} k_e (|X_e| + 1)e \]

単体の頂点集合としてのEはアフィン独立なので、H, H' ⊆ Eに対してそれらが張る面をH, H' とすれば、もし H, H' がその境界上にない1点を共有すれば、実は H と H' は集合としても一致すると言えるので、

\[ \{v\} \cup \left( \bigcup_{e \in R'} X_e : e \in R' \right) = R' \]

とわかる。特にこれから、各 e ∈ R' で X_e ∪ e ⊆ R' であるとわかる。

一方、全体集合がE ∈ Fなのでここで仮定1が必要？）あるfeasible set A ∈ Fがあって, |A ∩ R'| = 1となる。そこで {e'} = A ∩ R' とすると、上に述べたことから X_e ∩ A = ∅ となって、矛盾する。□

Lemma 9.13 ([102]) 点v ∈ Eに対してA ⊆ E \ v, R' ⊆ Rが存在してvがconv(A ∪ R')の内点であるとき、v ∈ τ(A)である。ただし τはアンチマトロイドFの閉包作用素。

（Proof）仮定からvは、正係数の頂点集合で

\[ \sum_{a \in A} k_a a + \sum_{r \in R'} p_r r = v \]

のように書ける。r = (|X_r| + 1)e_r - \sum(x : x ∈ X_r)としてこれを代入すると、

\[ \sum_{a \in A} k_a a + \sum_{r \in R'} p_r (|X_r| + 1)e_r = \sum_{r \in R'} p_r (\sum_{x : x \in X_r}) + v \]

再び頂点集合Eのアフィン独立性から

\[ A \cup \{e_r : r ∈ R'\} = \left( \bigcup \{X_r : r \in R'\} \right) \cup \{v\} \]

が成立する。ここで

\[ E' = \{e_r : r \in R'\}, \]
\[ X' = \bigcup \{X_r : r \in R'\}, \]
\[ A' = X' \setminus E', \]

とおくと、A' ⊆ Aかつv ∈ τ(A')である。

それ故するにはY ∈ F, Y ∩ A' = ∅ ならば常にv \notin Yを示せばよさ。逆に上の条件をみたすYがあってv ∈ Yであったとする。vが内点であるという仮定からv ∈ Aで、これよりv \notin A'。ゆえにv ∈ E'。ゆえに、E' ∩ Y ≠ ∅。ゆえにあるY' ⊆ Y, Y' ∈ Fがあって|Y' ∩ E'| = 1。
そこで $Y' \cap E' = \{ e_t \} \ (t \in R')$ とおく。定義から $X' \cup E' = A' \cup E'$。一方、仮定から $Y' \cap A' = \emptyset$。ゆえに

$$Y' \cap (X' \cup E') = Y' \cap (A' \cup E') = Y' \cap E' = \{ e_t \}$$

自明に $X_t \cup e_t \subseteq X' \cup E'$ なので $Y' \cap (X_t \cup e_t) = \{ e_t \}$. これは $F'$ のサーキットの性質に矛盾する。

ここで $A' \subseteq A$ だから $v \in \tau(A)$. □

Theorem 9.14 (Kashiwabara, Okamoto and Nakamura [102]) $R$ を核とする根付き凸シェリングアンチマトロイド $F_R$ は、もとのアンチマトロイド $F$ に一致する。

（Proof）まず $F \subset F_R$ を示す。$(X,e) \in C(F_R)$ とする。すると凸シェリングの閉包の定義から $e$ は $\text{conv}(X \cup R)$ の内点である。補題 2 から $e \in \tau(X)$ である。定義からある $X'' \subseteq X$ があって $(X'', e) \in C(F) \subseteq C(F_R)$ である。$F_R$ でのサーキットの極小性から $X = X''$ でありかつ $(X,e) \in C(F)$ とわたった。ゆえに、$C(F_R) \subseteq C(R)$ が示せ、これより $F \subseteq F_R$ と分かった。

その逆の $F_R \subseteq F$ を示そう。そこで $F_R$ のシェリング列の中で $L(F)$ に属さないものをその性質に関して極端なものをとり、それを $\alpha = e_1 e_2 \cdots e_m \in L(F_R)$ とする。$A = \{ e_1, \ldots , e_m \}$ とおく。$F_R \not\subseteq F$ なので $m < n = |E|$ である。すると $\alpha e \in L(F_R)$, $\alpha e \not\in L(F)$ となる元 $e \in E$ が存在する。これは、$e$ がアンチマトロイド $F_R/A$ の端点で $\text{conv}((E \setminus A) \cup R)$ の頂点であることを意味する。

一方、$A \cup e \not\in F$ より、ある $(X,e) \in C(F)$ があって $e \in A$ かつ $A \cap X = \emptyset$, i.e. $X \subseteq E \setminus A$ となる。ここで、$r = r(X,e)$ の定義から $e$ は $\text{conv}(X \cup r)$ の内点になり、ゆえに $\text{conv}((E \setminus A) \cup R)$ の内点になり、矛盾する。 □

定理 9.14 は、任意の凸幾何・アンチマトロイドがある kernelled affine convex geometry として表現できることを示しており、有向マトロイドの pseudoline による表現定理にも比すべき、凸幾何の表現定理と言える。
Chapter 10

Broken Circuit Complexes of Convex Geometries

10.1 Interior-point theorem and 1-sum of convex geometries

For a convex geometry \((K, \sigma, E)\), an element in \(\partial(E) = \bigcup \{\sigma(X) : X \subset \text{ex}(E)\}\) is a boundary point, and an element in \(\text{int}(E) = E - \partial(E)\) is called a relatively interior point.

A most typical example of a convex geometry arises from a point configuration in an affine space. Klain [107] and by Edelman and Reiner [65] independently proved a formula which counts the number of the relatively interior points of an affine point configuration.

In our term, they showed that the number of the relatively interior points is equal to the absolute value of the \(\beta\)-invariant of the affine convex geometry.

Let \(A\) be a finite set of points in \(\mathbb{R}^d\) whose convex hull, denoted by \(\text{conv}(A)\), is a polytope of dimension \(d\). Let \(\text{int}(A)\) be the set of points of \(A\) that lie in the interior region of \(\text{conv}(A)\). The collection of those subsets \(X\) which satisfy \(\text{conv}(X) \cap A = X\) constitutes a convex geometry, which we denote by \((A, K(A))\).

Then

**Theorem 10.1 ([65, 107])**

\[
|\text{int}(A)| = (-1)^{d+1} \sum_{X \in \text{Free}(K(A))} (-1)^{|X|} |X| \tag{10.1}
\]

It follows from this theorem and Theorem 7.15 that

**Corollary 10.2**

\[
|\text{int}(A)| = (-1)^{d+1} \beta(K(A)) \tag{10.2}
\]

Edelman and Reiner [65] also observed that the \(\beta\)-invariant of a convex geometry of a chordal graph is equal to the number of its one-vertex cuts, and the \(\beta\)-invariant of the double shelling of a poset agrees with the number of its bottlenecks. These observations can be generalized to our Theorem 10.4.

As for a loop-free convex geometry \((E, K)\), an element \(p \in E\) is a 1-cut if \(\{p\} \in K\) and the number of connected components of \(K \setminus p\) is greater than that of \(K\).
Suppose \((E_1, K_1)\) and \((E_2, K_2)\) to be a pair of loop-free convex geometries with \(E_1 \cap E_2 = \{p\}\) where \(|E_1|, |E_2| \geq 2\). And let us denote

\[
K_1^p \vee K_2^p = \{X \cup Y : X \in K_1^p, Y \in K_2^p\} \tag{10.3}
\]

where \(K_1^p = \{X \in K_1 : p \in X\}\) and \(K_2^p = \{Y \in K_1 : p \in Y\}\). Let us define

\[
K_1 \uplus K_2 = K_1 \cup K_2 \cup (K_1^p \vee K_2^p). \tag{10.4}
\]

It is a routine to check that \((E_1 \cup E_2, K_1 \uplus K_2)\) forms a convex geometry, which we shall call a 1-join of \((E_1, K_1)\) and \((E_2, K_2)\) at \(p\). In particular, if \(E_1 \setminus p \neq \emptyset, E_2 \setminus p \neq \emptyset\) and \(p\) is a coloop of both of \(K_1\) and \(K_2\), then a 1-join \((E_1 \cup E_2, K_1 \uplus K_2)\) is called a 1-sum.

**Theorem 10.3** Suppose \(K = K_1 \uplus K_2\) to be a 1-join at \(p\).

1. If \(p\) is a coloop of \((E_1, K_1)\) and \((E_2, K_2)\), then \(K \setminus p = (K_1 \setminus p) \oplus (K_2 \setminus p)\).
2. \(K/p\) is necessarily decomposed into a direct sum as \(K/p = (K_1/p) \oplus (K_2/p)\).

(Proof) The proof is an easy exercise. \(\square\)

**Theorem 10.4** Let \(K_1 \uplus K_2\) be a 1-sum at \(p\). Then,

1. \(p\) is a 1-cut of \(K_1 \uplus K_2\).
2. The \(\beta\)-invariant of \(K_1 \uplus K_2\) is designated as

\[
\beta(K_1 \uplus K_2) = \beta(K_1) + \beta(K_2) + 1 \tag{10.5}
\]

(Proof) (1) is obvious. To prove (2), let \(\sigma\) be the closure operator of \(K\), and \(\sigma_1, \sigma_2\) be the closure operator of \(K_1, K_2\), respectively. From (11.25) of Theorem 11.16, we have

\[
\beta(K_1 \uplus K_2) = \sum_{A \subseteq E_1 \cup E_2} (-1)^{|A|} |\sigma(A)| \tag{10.6}
\]

\[
= \sum_{X \subseteq E_1} (-1)^{|X|} |\sigma_1(X)| + \sum_{Y \subseteq E_2} (-1)^{|Y|} |\sigma_2(Y)| - \sum_{X = \emptyset, \{p\}} (-1)^{|X|} |\sigma(X)|
\]

\[
+ \sum_{X' \subseteq E_1, Y' \subseteq E_2} (-1)^{|X' \cup Y'| + |p|} |\sigma_1(X' \cup p) \cup \sigma_2(Y' \cup p)| \tag{10.7}
\]

\[
= \beta(K_1) + \beta(K_2) - \sum_{X = \emptyset, \{p\}} (-1)^{|X|} |\sigma(X)|
\]

\[
+ \sum_{X' \subseteq E_1, Y' \subseteq E_2} (-1)^{|X'| + |Y'| + 1} \left(|\sigma_1(X' \cup p)| + |\sigma_2(Y' \cup p)| - 1\right) \tag{10.8}
\]

Since \(K_1\) and \(K_2\) are loop-free, we have \(\sigma(\emptyset) = \sigma_1(\emptyset) \cup \sigma_2(\emptyset) = \emptyset\). From the assumption of \(\{p\} \in K_1\) and \(\{p\} \in K_2\), we have \(\{p\} \in K_1 \uplus K_2\), and so \(\sigma(\{p\}) = \{p\}\). Hence, the third term of (10.8) is equal to \(-1\).
Let us figure out the last term of (10.8).

The last term of (10.8) =
\[
\left(\sum_{X' \subseteq E_1'} (-1)^{|X'|} \sigma_1(X' \cup p)\right) \left(\sum_{Y' \subseteq E_2'} (-1)^{|Y'|} \sigma_2(Y' \cup p)\right)
\]
\[
+ \left(\sum_{X' \subseteq E_1'} (-1)^{|X'|} \right) \left(\sum_{Y' \subseteq E_2'} (-1)^{|Y'|+1} \right)
\]
\[
+ \left(\sum_{X' \subseteq E_1'} (-1)^{|X'|} \right) \left(\sum_{Y' \subseteq E_2'} (-1)^{|Y'|} \right)
\]
\[
(10.9)
\]

Since \(E_1' \neq \emptyset\) and \(E_2' \neq \emptyset\) are assumed, we have
\[
\sum_{X' \subseteq E_1'} (-1)^{|X'|} = \sum_{Y' \subseteq E_2'} (-1)^{|Y'|} = 0.
\]
\[
(10.10)
\]

Hence the last term of (10.8) is equal to 0. Thus we have proved
\[
\beta(K_1 \upharpoonright K_2) = \beta(K_1) + \beta(K_2) + 1
\]
\[\Box\]

10.2 Brylawski decomposition of broken-circuit complexes of convex geometries

続いて、凸庁何の Orlik-Solomon 代数を定義する。まず、前記と同様に \(e_1, \ldots, e_n\) を生成元とする free module \(\oplus_{e \in \mathcal{E}} \mathbb{Z}e\) 上の graded external algebra を \(\wedge E = \oplus_{i \in \mathbb{N}} \wedge^i E\) とする。以下、\(K\) は \(E\) 上の loop-freeな凸庁何とする。

ここで \(\{e_X : X \text{ is a broken circuit of } K\}\) によって生成されるイデアルを \(I_K\) としたとき、
\[
A(K) = \left(\wedge E\right) / I_K
\]
(10.11)

を、凸庁何 \(K\) の Orlik-Solomon algebra と呼ぶことにする。定義から自明に

Proposition 10.5 \(\{e_X : X \in BC(K)\}\) は module \(A(K)\) の linear basis になる。

元 \(x\) が凸庁何 \(K\) の coloop のとき、Theorem 10.10 の直和分解 (10.18) が成立立てるので、Proposition 10.5 から次の short exact split sequence theorem が導かれる。

Corollary 10.6 元 \(x\) が凸庁何 \(K\) の coloop のとき
\[
0 \rightarrow A(K \setminus x) \xrightarrow{i_x} A(K) \xrightarrow{p_x} A(K/x) \rightarrow 0
\]
(10.12)

は short exact split sequence である。
(定理 10.10 の証明) 証明の前に、補題を一つ用意する。任意の元 $x$ に対して、$K$ の broken circuit の全体 $BC(K) \cup e : e \in E$ を次の2つに分割する。

$$BC(K)^x = \{ S \in BC(K) : x \in S \} , \quad BC(K)_x = \{ S \in BC(K) : x \notin S \} \quad (10.13)$$

このとき、

**Lemma 10.7** $x$ ハ coloop であるとすると、$BC(K)^x = BC(K \setminus x)$。

(Proof) $T \in BC(K)^x$ と仮定する。定義からある $K$ の根付きサーキット $(T, e)$ が存在して $x \notin T$ である。$x$が coloop だから必ず $e \neq x$ である。ここで $e \in \sigma(T) = \sigma^x(T)$ でかつこの性質に関して極小だから $(T, e)$ は $K \setminus x$ の根付きサーキットである。

逆に、$T \in BC(K \setminus x)$ とする。定義からある元 $e \neq x$ が存在して $(T, e)$ が $K \setminus x$ の根付きサーキットになる。上と同様の議論から $(T, e)$ が $K$ の根付きサーキットであると分かる。ゆえに、$T \in BC(K)^x$。ゆえに、$BC(K)^x = BC(K \setminus x)$ と分かった。

Section 6.5 で述べたマトライドでの諸結果にアナログカルに対応する凸幾何での諸結果を順に紹介する。

まず、凸幾何における Whitney-Rota の公式を証明する。$K$ を有限非空集合 $E$ 上の凸幾何とする。凸幾何の閉集合族 $K$ は meet-distributive lattice になる。凸幾何 $K$ をこの束とみなしたときの対応する特徴多項式は

$$p(K; \lambda) = \sum_{X \in K} \mu_K(\emptyset, X) \lambda^{|E| - |X|} \quad (10.14)$$

である。ここでは簡単のために $\emptyset \in K$ と仮定しておく。

Edelman [61] は、凸幾何 $K$ での Möbius function $\mu_K$ の値を explicit に計算して示した。それを書き換えると、以下になる。

**Proposition 10.8 ([61])**

$$p(K; \lambda) = \sum_{X \in free(K)} (-1)^{|X|} \lambda^{|E| - |X|} \quad (10.15)$$

**Theorem 7.15** $Free(K) = BC(K)$ であると分かったので、これから直ちに、

**Corollary 10.9 (Whitney-Rota’s formula for convex geometries)**

$$p(K; \lambda) = \sum_{X \in BC(K)} (-1)^{|X|} \lambda^{|E| - |X|} \quad (10.16)$$

次に、凸幾何での deletion, contraction の定義を示し、Brylawski の分解定理のアナログーが凸幾何でも成立することを示す。

$Ex(E)$ の元を coloop と呼ぶことにする。$e$ が coloop であるとき、$K \setminus e = \{ X : X \in K, e \notin X \}$ は $E \setminus e$ 上の凸幾何になる。これを $K$ から $e$ を delete したものである。任意の元 $e \in E$ に対して 10.16 で示したもののと呼ぶ。$K$ から deletion と contraction を繰り返して得られる凸幾何を $K$ のマイナーと呼んでいる。

$K \setminus e$ と $K/e$ の closure operator を $\sigma^e, \sigma_e$ とするとき

$$\sigma^e(A) = \sigma(A), \quad \sigma_e(A) = \sigma(A \cup e) \setminus e \quad (A \subseteq E \setminus e) \quad (10.17)$$

が成立立っている。ただしここで $\sigma$ は $K$ の closure operator である。

次が、Brylawski の分解定理に対応する凸幾何での broken circuit complex の分解定理である。
Theorem 10.10 元 $x \in E$ が凸幾何 $K$ の coloop であるとすると、

$$BC(K) = BC(K \setminus x) \cup (BC(K/x) \ast x)$$

(10.18)

ただしここで、$BC(K/x) = \emptyset$ の場合は、$BC(K/x) \ast x = \emptyset$ とおくことにする。

( Theorem 10.10 の証明は、このセクションの末尾に記しておく。) (Proof of Theorem 10.10) まず $BC(K)^x = \{ A : x \not\in A, A \in BC(K) \}$, $BC(K)_x = \{ A : x \in A, A \in BC(K) \}$ とおくと、自明に

$$BC(K) = BC(K)^x \cup BC(K)_x$$

は $BC(K)$ の分割になっている。そこで、定理を示すには以下

(i) $BC(K)^x = BC(K \setminus x)$,
(ii) $BC(K)_x = BC(K/x) \ast x$.

を示せばよい。 (八森君のコメントから：上の i), ii) は Edelman, Reiner and Welker [66] の Lemma 17 に同値です。ただし、$BC$ complex を Free complex に言い換えた形になっています。

(i) の証明: $A \subseteq E \setminus x$ とする。ここで $BC(K) = BC(K)^x \cup BC(K)_x$ だが $x \not\in A$ なので、$A$ が $BC(K)_x$ の元を含まないことは自明。ゆえに、$A \in BC(K)$ であるための必要十分条件は、$A$ が $BC(K)^x$ の元を含まないことである。ここで、Lemma 10.7 から $BC(K)^x = BC(K \setminus x)$ であるから、$A \in BC(K)$ であるための必要十分条件は、$A$ が $BC(K \setminus x)$ の元を含まないことであり、言い換えれば、$A \in BC(K \setminus x)$ である。これで (i) が示せた。

(ii) の証明: (ii) を示すには、$N' = \{ X - x : X \in BC(K)_x \}$ とおいて、$N' = BC(K/x)$ を示せばよい。それには、

a) $A \in N'$ ならば $A \in BC(K)x$.

b) $A \in BC(K)x$ ならば $A \in N'$

を示せばよい。

a) の証明: 仮定から $A \cup x = BC(K)$. ここで逆に $A \not\in BC(K/x)$ であったと仮定してみる。すると、$K/x$ の根付きサーキット $(T, e)$ で $T \subseteq A$ なるものがあることがある。根付きサーキットの性質から $e \in \sigma_x(T) = \sigma(T \cup x)$ であり、$e \neq x$ だから $e \in \sigma(T \cup x)$ である。ゆえに、ある $T' \subseteq T \cup x$ が存在して $(T', e)$ が $K$ の根付きサーキットになる。ゆえに $T' \subseteq T \cup x \subseteq A \cup x$ であったので、これは $A \cup x \in BC(K)$ で $A \cup x$ が $K$ の broken circuit を含まないとした前提に矛盾する。ゆえに、必ず $A \in BC(K/x)$ と分かった。

b) の証明: $A \in BC(K/x)$ とする。ここで逆に $A \not\in N'$ つまり $A \cup x \not\in BC(K)$ と仮定する。すると $A$ の根付きサーキット $(S, e)$ があって、$S \subseteq A \cup x$ を満たす。定義から $e \in \sigma(S) \subseteq \sigma(A \cup x)$ で、一方 $e \neq x$ だから、$e \in \sigma(A \cup x) \setminus x = \sigma_x(A)$ と分かる。これは直ちに $A$ は $K/x$ の broken circuit を意味するが、これは $A \in BC(K/x)$ の仮定に矛盾する。ゆえに、$A \cup x \in BC(K)$ と分かって、b) が証明された。

これで (ii) の証明が終わり、かつ全体の証明が終わった。
Chapter 11

Tutte Polynomials and \( \beta \)-variants of Closure Systems

11.1 Rank generating functions and \( \beta \)-invariants of closure systems

For a nonempty finite set \( E \), we shall call a non-negative valued function \( r : 2^E \rightarrow \mathbb{Z} \) a rank function provided \( 0 \leq r(A) \leq |A| \) for any \( A \subseteq E \). Formally we can define a generating function \( R(r; x, y) \) as below.

\[
R(r; x, y) = \sum_{A \subseteq 2^E} x^{r(E)} y^{|A| - r(A)}
\]

(11.1)

\[
T(r; x, y) = R(r; x - 1, y - 1)
\]

(11.2)

As an analogue of the case of matroids, the \( \beta \)-invariant can be formally defined.

\[
\beta_r = \frac{\partial R}{\partial x} \bigg|_{x = y = -1} = (-1)^{r(E)} \sum_{A \subseteq 2^E} (-1)^{|A|} r(A)
\]

(11.3)

For a nonempty finite set \( E \), we shall call a non-negative valued function \( r : 2^E \rightarrow \mathbb{Z} \) a rank function provided \( 0 \leq r(A) \leq |A| \) for any \( A \subseteq E \). Formally we can define a generating function \( R(r; x, y) \) as below.

\[
R(r; x, y) = \sum_{A \subseteq 2^E} x^{r(E)} y^{|A| - r(A)}
\]

(11.1)

\[
T(r; x, y) = R(r; x - 1, y - 1)
\]

(11.2)

As an analogue of the case of matroids, the \( \beta \)-invariant can be formally defined.

\[
\beta_r = \frac{\partial R}{\partial x} \bigg|_{x = y = -1} = (-1)^{r(E)} \sum_{A \subseteq 2^E} (-1)^{|A|} r(A)
\]

(11.3)

Let \( (K, \sigma, E) \) be a loop-free closure space. Let \( O \) be the open-set family which is the complement of \( K \).

For the moment, it seems that we have two possible choices of rank functions for closure systems.

\[
r_{\text{ex}}(A) = |\text{ex}(A)| \quad \text{for } A \subseteq E,
\]

(11.4)

\[
r_{\text{cl}}(A) = \max\{|A| : A \in O\} = |E| - |\sigma(E - A)| \quad \text{for } A \subseteq E.
\]

(11.5)

Each rank function gives a rank generating function \( R_{\text{ex}}(x, y) = R(r_{\text{ex}}; x, y) \) and \( R_{\text{cl}} = R(r_{\text{cl}}; x, y) \). Correspondingly, we have the Tutte polynomials \( T_{\text{ex}} \) and \( T_{\text{cl}} \), and the \( \beta \)-invariants \( \beta_{\text{ex}} \) and \( \beta_{\text{cl}} \).

**Example 11.1** \( T_{\text{ex}} \) and \( T_{\text{cl}} \) are generally not equal. For instance, suppose that a matroid derived from three colinear in an affine space is given, and consider the closure operator and the closure system of flats of the matroid. Then

\[
T_{\text{ex}}(x, y) = x^2 + x + y - 1,
\]

(11.6)

\[
T_{\text{cl}}(x, y) = x^3 y - 3x^2 y + 3xy + 2x^2 - x + y
\]

(11.7)
Then the following hold.

**Proposition 11.4** Let based on the extreme-point rank function $r_{ex}$, the rank generating function $R_{ex}$ have the Tutte-Grothendieck type properties or not.

1. Let $\beta$ be the associated rank generating function with a closure system $(K, E)$, then

$$\beta_{cl} = \frac{\partial T_{cl}}{\partial x} \bigg|_{x=y=0} = (-1)^{|\sigma(\emptyset)|-1} \sum_{A \in 2^E} (-1)^{|A|} |\sigma(A)|$$

We shall investigate whether the rank generating functions and the Tutte functions of closure systems have the Tutte-Grothendieck type properties or not.

1. The rank generating function $R_{ex}$ defined from the extreme-point rank function $r_{ex}$ satisfies the following, but does not enjoy the Tutte-Grothendieck type recursion relation.

**Proposition 11.2** Let $R_{ex}(K)$ be the associated rank generating function with a closure system $K$, defined based on the extreme-point rank function $r_{ex}$.

1. $R_{ex}(K_1 \oplus K_2) = R_{ex}(K_1) R_{ex}(K_2)$.
2. $R_{ex}(\emptyset, \emptyset) = 1$, $R_{ex}(\text{a coloop}) = x + 1$, $R_{ex}(\text{a loop}) = y + 1$.

**Corollary 11.3**

1. If $K$ is disconnected, then $\beta_{ex} = 0$.
2. $\beta_{ex}(\text{a coloop}) = 1$, $\beta_{ex}(\text{a loop}) = 0$.

Next we shall consider $R_{cl}$ arising from the closure rank function $r_{cl}$, and the $\beta$-invariant $\beta_{cl}$.

**Proposition 11.4** Let $R_{cl}(K)$ be the corresponding rank generating function of a closure system $(K, E)$. Then the following hold.

1. $R_{cl}(K_1 \oplus K_2) = R_{cl}(K_1) R_{cl}(K_2)$.
2. $R_{cl}(\emptyset, \emptyset) = 1$, $R_{cl}(\text{a coloop}) = x + 1$, $R_{cl}(\text{a loop}) = y + 1$.
3. If $e$ is a coloop, i.e. $e \in \text{ex}(E)$, then

$$R_{cl}(K) = x R_{cl}(K/e) + R_{cl}(K \setminus e)$$
$$T_{cl}(K) = (x-1) T_{cl}(K/e) + T_{cl}(K \setminus e)$$

4. If there is no coloop for $K$, i.e. $K = \{E\}$, then

$$R_{cl}(x, y) = (1 + y)^{|E|}, \quad T_{cl}(x, y) = y^{|E|}.$$
(2) 定義から明らか。
(3) 式 (11.8) の rank generating function が (3) の recursion relation をみたすことを示す。$K/e$ の閉包関数を $\sigma_e, K \setminus e$ の閉包関数を $\sigma^e$ とすると、$\sigma_e(X) = \sigma(X \cup e) \setminus e$. $e$ が端点だから $|\sigma(X \cup e)| = |\sigma_e(X)| + 1$ ($X \subseteq E' = E \setminus e$)、$e$ がループでないので $\sigma(\emptyset) = \sigma_e(\emptyset)$. また、$\sigma^e(X) = \sigma(X)$ ($X \subseteq E' = E \setminus e$) は明らかなので、式 (11.8) は次のように計算できる。

$$R_{\text{cl}}(K) = \sum_{A \subseteq E, c \in A} x^{|\sigma(A)| - |\sigma(\emptyset)|} y^{|\sigma(A)| - |A|} + \sum_{A \subseteq E, c \notin A} x^{|\sigma(A)| - |\sigma(\emptyset)|} y^{|\sigma(A)| - |A|}$$

$$= \sum_{X \subseteq E'} x^{|\sigma(X \cup e)| - |\sigma(\emptyset)|} y^{|\sigma(X \cup e)| - |X \cup e|} + \sum_{X \subseteq E'} x^{|\sigma^e(X)| - |\sigma(\emptyset)|} y^{|\sigma^e(X)| - |X|}$$

$$= \sum_{X \subseteq E'} x^{|\sigma_e(X)| + 1 - |\sigma_e(\emptyset)|} y^{|\sigma_e(X)| + 1 - (|X| + 1)} + R_{\text{cl}}(K \setminus e)$$

$$= x R(K/e) + R(K \setminus e)$$

(Proof of (3)) Since $\sigma(A) = E$ for any $A \subseteq E$, we have

$$R_{\text{cl}}(x, y) = \sum_{A \subseteq E} x^{|\sigma(A)| - |\sigma(\emptyset)|} y^{|\sigma(A)| - |A|} = \sum_{A \subseteq E} y^{|E| - |A|} = (1 + y)^{|E|}.$$  

□

The following corollary is immediate from the above argument.

**Corollary 11.5** For a closure system $(E, K)$, the following hold.

(1) If $K$ is disconnected, then $\beta_{\text{cl}} = 0$.

(2) $\beta_{\text{cl}}(a coloop) = 1$, $\beta_{\text{cl}}(a loop) = 0$

(3) If $e$ is a coloop, then

$$\beta_{\text{cl}}(K) = -\beta_{\text{cl}}(K/e) + \beta_{\text{cl}}(K \setminus e)$$

( Gordon and McMahon [83] referred to this fact in case of chordal graphs.)

(Proof) (1) and (2) follow from the definitions. (3) is established by the following calculation.

$$\frac{\partial R(K)}{\partial x}(-1, -1) = \frac{\partial x R(K/e)}{\partial x}(-1, -1) + \frac{\partial R(K \setminus e)}{\partial x}(-1, -1)$$

$$= \left(x \frac{\partial R(K/e)}{\partial x}\right)(-1, -1) + R(K/e)(-1, -1) + \frac{\partial R(K \setminus e)}{\partial x}(-1, -1)$$

$$= -\beta_{\text{cl}}(K/e) + \beta_{\text{cl}}(K \setminus e)$$  

□

For a closure space $(K, \sigma, E)$, $T_{\text{ex}}(x, y)$ has the following properties.

$$T_{\text{ex}}(0, 0) = R_{\text{ex}}(-1, -1) = 0, \quad T_{\text{ex}}(2, 2) = R_{\text{ex}}(1, 1) = |2^E|.$$  

$$T_{\text{ex}}(2, 1) = R_{\text{ex}}(1, 0) = |\{A \subseteq E : |E(A)| = |A|\}| = The \ number \ of \ independent \ sets.$$  

(11.10)  (11.11)

As is easy to see as well, $T_{\text{cl}}(x, y)$ satisfies the following.

$$T_{\text{cl}}(0, 0) = R_{\text{cl}}(-1, -1) = 0, \quad T_{\text{cl}}(2, 2) = R_{\text{cl}}(1, 1) = |2^E|.$$  

$$T_{\text{cl}}(2, 1) = R_{\text{cl}}(1, 0) = |\{A \subseteq E : \sigma(A) = A\}| = The \ number \ of \ closed \ sets.$$  

(11.12)  (11.13)

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11.2 Characteristic polynomials and $\beta$-invariants of convex geometries

Throughout this section, we assume $(E, K)$ to be a loop-free convex geometry. According to the definitions (4.8) and (4.9), we have a characteristic polynomial $p(K; \lambda)$ and a $\beta$-invariant $\beta(K)$ for the convex geometry. They are defined as

$$p(K; \lambda) = \sum_{X \in K} \mu_K(\emptyset, X)\lambda^{|E|-|A|}$$

(11.14)

$$\beta(K) = \sum_{X \in K} \mu_K(\emptyset, X)|X|$$

(11.15)

The fundamental properties (I), (II), (III) and (IV) of the characteristic polynomials and the $\beta$-invariants of matroids have nice analogues in convex geometries, which we shall describe below sequentially.

(I) A decomposition of NBC complexes:

The NBC complex of a convex geometry has a decomposition theorem which is completely analogous to Brylawski’s decomposition (6.7) of matroids.

Theorem 11.6 (Brylawsky [37])  For a cooop $x$ of a convex geometry $K$, the NBC complex is decomposed into a direct sum as

$$NBC(K) = NBC(K \setminus x) \cup (NBC(K/x) \ast x)$$

(11.16)

where $K \setminus x = \{X \in K : e \notin K\}$ and $K/x = \{X \setminus x : X \in K, e \in X\}$.

We first prove preparatory lemmas. For each element $x$, obviously $NBC(K)$ is partitioned into $NBC(K)^x = \{X \in NMC(K) : x \notin X\}$ and $NBC(K)x = \{X \in NMC(K) : x \in X\}$.

Lemma 11.7  If $x$ is a cooop, the $NBC(K)^x = NBC(K \setminus x)$.

(Proof) Suppose $T \in NBC(K)^x$. By definition, there is a rooted circuit $(T, e)$ of $K$. Since $x$ is a cooop, we have $x \neq e$. Obviously $e\sigma(T) = \sigma^x(T)$, and $T$ is minimal with respect to this property. Hence $(T, e)$ is a rooted circuit of $K \setminus x$.

Conversely suppose $T \in NBC(K \setminus x)$. By definition there exists an element $e \neq x$ such that $(T, e)$ is a rooted circuit of $K$. It follows the similar argument above that $(T, e)$ is a rooted circuit of $K$. Hence we have $T \in NBC(K)$. This completes the proof. \qed

Lemma 11.8  $NBC(K)x = NBC(K/x) \ast x$.

(Proof) Let $N' = \{X \setminus x : X \in NBC(K)x\}$. We only have to show $N' = NBC(K/x)$.

Suppose $A \in N'$. This implies $A \cup x \in NBC(K)$. Now suppose contrarily that $A \notin NBC(K/x)$. Then there exists a rooted circuit $(T, e)$ of $K/e$ such that $T \subseteq A$. By 5.7, we have $e \in \sigma_x(T) = \sigma(T \cup x) \setminus x$. From $e \neq x$, $x\sigma(T \cup x)$ follows. Hence there exists $T' \subseteq T \cup x$ such that $(T', e)$ is a rooted circuit of $K$. This implies that $T'$ is a broken circuit of $K$ with $T' \subseteq T \cup x \subseteq A \cup x$, which contradicts the assumption $A \cup x \in NBC(K)$. This proves $A \in NBC(K/x)$.

Suppose $A \in NBC(K/x)$. And suppose contrarily that $A \notin N'$, i.e., $A \cup x \notin NBC(K)$. This implies that there is a rooted circuit $(S, e)$ of $K$ such that $S \subseteq A \cup x$. Hence we have $e \in \sigma(S) \subseteq \sigma(A \cup x)$. \qed
This shows $e \in \sigma(A \setminus x) \setminus x = \sigma_x(A)$. Hence $A$ contains a broken circuit of $K/x$, which contradicts the assumption $A \in NBC(K/x)$. This proves $A \cup x \in NBC(K)$.

This completes the proof of the lemma. □

(Proof of Theorem 11.6) Since $NBC(K) = NBC(K)^x \cup NBC(K)_x$ is trivially a partition of $NBC(K)$. Hence Theorem directly follows from Lemmas 11.7 and 11.8. □

(II) Expansions of characteristic polynomials over NBC complexes:

Next we shall show the expansions of characteristic polynomials and $\beta$-invariants over NBC complexes.

The values of the Möbius function of a convex geometry are known.

Lemma 11.9 ([63]) For a convex geometry $K$ and $X, Y \in K$, the interval $[X, Y]$ is a Boolean algebra if and only if $Y - X \subseteq ex(Y)$.

(Proof) If $X \subseteq Y$ and $Y - X \subseteq ex(Y)$, then $[X, Y] \cong 2^{|Y| - |X|}$. We show the necessity part.

Let $Y_1, \ldots, Y_k$ be the elements covered by $Y$. There exist $k$ distinct elements $z_1, \ldots, z_k$ such that $Y_i = Y - z_i \in K$ for $i = 1, \ldots, k$ as $K$ is a convex geometry. Then each $z_i$ is in $ex(Y)$. Then clearly $\bigcap_i Y_i = Y - \{z_1, \ldots, z_k\}$. Since $[X, Y]$ is a Boolean algebra, $X = \bigcap_i Y_i = Y - \{z_1, \ldots, z_k\}$. This implies $Y - X = \{z_1, \ldots, z_k\} \subseteq ex(Y)$. This completes the proof. □

Proposition 11.10 ([63]) For a convex geometry $K$ and $X, Y \in K$ with $X \subseteq Y$, The value of the Möbius function is given by

$$\mu(X, Y) = \begin{cases} (-1)^{|Y| - |X|} & \text{if } Y - X \subseteq ex(Y) \\ 0 & \text{otherwise.} \end{cases} \quad (11.17)$$

(Proof) The cross-cut theorem in [144] says that if $X$ is not the meet of the elements covered by $Y$, then $\mu(X, Y) = 0$. Contrarily if $X$ is the meet of the elements covered by $Y$, $[X, Y]$ is a Boolean algebra as a convex geometry is meet-distributive. The value of the Möbius function of a Boolean algebra of the height $k$ is $(-1)^k$. Hence in the latter case, $\mu(X, Y) = (-1)^{|Y| - |X|}$ □

Corollary 11.11 ([63]) Let $\mu_K$ be the Möbius function of the lattice $K$ of convex sets of a convex geometry. Then for each closed set $X \in K$, we have

$$\mu_K(\emptyset, X) = \begin{cases} (-1)^{|X|} & \text{if } X = ex(X), \text{ i.e. } X \text{ is free,} \\ 0 & \text{otherwise.} \end{cases} \quad (11.18)$$

Corollary 11.12 ([63]) Let $F_k$ be the number of free sets of cardinality $k$ in a convex geometry $K$. Then $\sum_k (-1)^k F_k = 0$.

(Proof) $\sum_k (-1)^k F_k = \sum_{\{X \in K : X \text{ is free} \}} \sum_X \mu_K(\emptyset, X) = \sum_X \mu_K(\emptyset, X) = 0.$

The last equality comes from the definition of the Möbius function.

This automatically leads to the following.
Proposition 11.13 The characteristic polynomial and the $\beta$-invariant of $(E, \mathcal{K})$ are described as

\[
p(K; \lambda) = \sum_{X \in \text{Free}(K)} (-1)^{|X|} \lambda^{|E|-|X|}, \quad (11.19)
\]

\[
\beta(K) = \sum_{X \in \text{Free}(K)} (-1)^{|X|} |X|. \quad (11.20)
\]

Since $\text{Free}(K)$ is equal to $\text{NBC}(K)$ from Theorem 7.15, we have

Theorem 11.14 (Whitney-Rota’s formula for convex geometries)

\[
p(K; \lambda) = \sum_{X \in \text{NBC}(K)} (-1)^{|X|} \lambda^{|E|-|X|}, \quad (11.21)
\]

\[
\beta(K) = \sum_{X \in \text{NBC}(K)} (-1)^{|X|} |X|. \quad (11.22)
\]

We remark that the equation (11.21) is referred to in the discussion after Proposition 6 of [82].

(III) Boolean expansion:

In the following we shall state the Boolean expansions of characteristic polynomials and $\beta$-invariants.

It is easy to see that for each closed set $X \in \mathcal{K}$, $\tilde{X} = \{ \text{ex}(X), X \}_{2^E} = \{ A \subseteq E : \text{ex}(X) \subseteq A \subseteq E \}$ is a sublattice of $2^E$ and is itself a Boolean algebra. Hence we have

Lemma 11.15 For each closed set $X \in \mathcal{K}$, we have

\[
\sum_{A \in \tilde{X}} (-1)^{|A|} = \begin{cases} (-1)^{|X|} & \text{if } X = \text{ex}(X), \text{ i.e. } X \text{ is free}, \\ 0 & \text{otherwise}. \end{cases} \quad (11.23)
\]

On the other hand, it is an easy observation that the collection $\tilde{K} \equiv \{ \tilde{X} : X \in \mathcal{K} \}$ forms a partition of $2^E$. Then it follows from Lemma 11.15 and the remark above that

Theorem 11.16 The characteristic polynomial of a convex geometry $(E, \mathcal{K})$ can be exhibited by the Boolean expansions as below.

\[
p(K; \lambda) = \sum_{A \in 2^E} (-1)^{|A|} \lambda^{|E|-|\text{ex}(A)|} = \sum_{A \in 2^E} (-1)^{|A|} \lambda^{|E|-|\sigma(A)|} \quad (11.24)
\]

(Proof) The crucial point of the proof is that if $X$ is free then $X = \text{ex}(X) = \sigma(X)$. We can work out from Lemma 11.15 and the succeeding comment as

\[
\sum_{A \in 2^E} (-1)^{|A|} \lambda^{|E|-|\text{ex}(A)|} = \sum_{X \in \mathcal{K}} \left( \sum_{A \in \tilde{X}} (-1)^{|A|} \lambda^{|E|-|\text{ex}(A)|} \right) = \sum_{X \in \mathcal{K}} \lambda^{|E|-|\text{ex}(A)|} \left( \sum_{A \in \tilde{X}} (-1)^{|A|} \right) = \sum_{X \in \text{Free}(K)} \lambda^{|E|-|\text{ex}(X)|} (-1)^{|X|} = \sum_{X \in \text{Free}(K)} (-1)^{|X|} \lambda^{|E|-|X|}.
\]

Hence the first equality of (11.24) is proved. The second equality of (11.24) can be shown in the same way.

\[\square\]

The definition (11.15) of $\beta(K)$ and the expansion (11.24) above immediately imply

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Corollary 11.17

\[ \beta(K) = \sum_{A \in 2^E} (-1)^{|A|} ex(A) = \sum_{A \in 2^E} (-1)^{|A|} \sigma(A) \]  

(11.25)

(The second equality of (11.25) is shown in Gordon and McMahon \[82\].)

In general we can take an arbitrary integer-valued function \( r : 2^E \to \mathbb{Z} \) as a rank function, and define a rank generating function \( R(r; x, y) \) by

\[ R(r; x, y) = \sum_{A \in 2^E} x^{r(E)} y^{|A| - r(A)}. \]  

(11.26)

The corresponding characteristic polynomial \( p_R \) and the \( \beta \)-invariant \( \beta_R \) can be formally defined by

\[ p_R(r; \lambda) = (-1)^{r(E)} R(-\lambda, -1) = \sum_{A \in 2^E} (-1)^{|A|} \lambda^{r(E) - r(A)}, \]  

(11.27)

\[ \beta_R(r) = (-1)^{r(E)} \frac{\partial R(r; x, y)}{\partial x} (-1, -1) = \sum_{A \in 2^E} (-1)^{|A|} r(A). \]  

(11.28)

For a convex geometry, there are two apparently plausible choices of a rank function. For a set \( A \subseteq E \), the \textit{closure rank} of \( A \) is the cardinality of the maximum feasible set in \( A \), and we denote it by \( r_{cd}(A) \). The \textit{extremal rank} of \( A \) is the size of the smallest generating set in \( A \). We denote it by \( r_{ex}(A) \). Actually, \( Ex(A) \) is the unique generating set in \( A \). Hence,

\[ r_{cd}(A) = |E| - |\sigma(E \setminus A)|, \]  

(11.29)

\[ r_{ex}(A) = |Ex(A)|. \]  

(11.30)

Since \( r_{cd} \) and \( r_{ex} \) are not equal in general, the associated rank generating functions \( R(r_{cd}; x, y) \) and \( R(r_{ex}; x, y) \) are distinct. And in general the characteristic polynomials \( p_R(r_{cd}; \lambda) \) and \( p_R(r_{ex}; \lambda) \) are also distinct. For instance, suppose three points to lie on a line. For the convex geometry given by this affine configuration, we have \( R(r_{cd}; x, y) = x^3 + x^3 y + 2x^2 + 3x + 1 \) and \( p_R(r_{cd}; \lambda) = -(2\lambda - 1)(\lambda - 1) \), while \( R(r_{ex}; x, y) = x(x^3 + 3x + 3 + y) \) and \( p_R(r_{ex}; \lambda) = \lambda(\lambda - 1)(\lambda - 2) \) hold. However, (11.24) of Theorem 11.16 asserts that \( \beta_R(r_{cd}) \) and \( \beta_R(r_{ex}) \) are both equal to \( \beta(K) \). That is, we can restate (11.24) as

\[ \beta(K) = \beta_R(r_{ex}) = \beta_R(r_{cd}) \]  

(11.31)

(IV) The direct-sum factorizations and the deletion-contraction rules:

Theorem 11.18 (Gordon \[81\], Gordon and McMahon \[82\]) Let \((E, K)\) be a convex geometry.

1. If \((E, K)\) is loop-free and decomposed into a nontrivial direct sum \( K = K_1 \oplus K_2 \), then

\[ p(K; \lambda) = p(K_1; \lambda)p(K_2; \lambda), \]  

(11.32)

\[ \beta(K) = 0. \]  

(11.33)

2. If \( e \in E \) is a coloop of \( K \) and \( E \setminus \{e\} \neq \emptyset \), then

\[ p(K; \lambda) = -p(K/e; \lambda) + \lambda p(K \setminus e; \lambda), \]  

(11.34)

\[ \beta(K) = -\beta(K/e) + \beta(K \setminus e). \]  

(11.35)
(Proof)  

(1) Let $\sigma_1, \sigma_2$ denote the closure operator of $\mathcal{K}_1, \mathcal{K}_2$, respectively, and $\sigma$ denote that of $\mathcal{K}$. Then from Proposition 5.18, for any subset $A \subseteq E$, we have $\sigma(A) = \sigma(A \cap E_1) \cup \sigma(A \cap E_2)$ (disjoint union). Based on this, we can figure out (11.24) as

$$p(\mathcal{K}; \lambda) = \sum_{A \subseteq E} (-1)^{|A|} |E| - |\sigma(A)| = \sum_{A \subseteq E_1} \sum_{A_2 \subseteq E_2} (-1)^{|A_1| + |A_2|} |(E_1 + E_2)| - (|\sigma(A_1)| + |\sigma(A_2)|)$$

$$= \left( \sum_{A \subseteq E_1} (-1)^{|A_1|} |E_1| - |\sigma(A_1)| \right) \left( \sum_{A_2 \subseteq E_2} (-1)^{|A_2|} |E_2| - |\sigma(A_2)| \right) = p(\mathcal{K}_1; \lambda)p(\mathcal{K}_2; \lambda)$$

Thus the factorization (11.32) is attained.

From the assumption that the direct sum is nontrivial, $p(\mathcal{K}_1; 1) = p(\mathcal{K}_2; 1) = 0$ follows. By definition and (11.32), we have

$$\beta(\mathcal{K}) = -\frac{dp(\mathcal{K}_1; \lambda)p(\mathcal{K}_2; \lambda)}{d\lambda}(1) = -p(\mathcal{K}_1; 1)\frac{dp(\mathcal{K}_2; \lambda)}{d\lambda}(1) - p(\mathcal{K}_2; 1)\frac{dp(\mathcal{K}_1; \lambda)}{d\lambda}(1) = 0.$$

(2) Let $\sigma_e$ denote the closure operator of $\mathcal{K}/e$. Then we have $|\sigma(A' \cup e)| = |\sigma_e(A')| + 1$ for $A' \subseteq E \setminus e$. Since $e$ is supposed to be a coloop of $(E, \mathcal{K})$, it is certain that $\mathcal{K} \setminus e$ is again a closed set family and $(E \setminus e, \mathcal{K} \setminus e)$ is a convex geometry. Let $\sigma^*$ denote the closure operator of $(E \setminus e, \mathcal{K} \setminus e)$. Then we have $|\sigma(A')| = |\sigma^*(A')|$ for $A' \subseteq E \setminus e$. Setting $E' = E \setminus e$ and making use of the expansion (11.24), we can work out as

$$p(\mathcal{K}; \lambda) = \sum_{A \subseteq E_2} (-1)^{|A|} |E| - |\sigma(A)| = \sum_{A \subseteq E_1} \sum_{A_2 \subseteq E_2} (-1)^{|A_1|} |E_1| - |\sigma(A_1)| + \sum_{A \subseteq E_2} (-1)^{|A_1|} |E_2| - |\sigma(A_1)|$$

$$= \sum_{A' \subseteq E'} (-1)^{|A'|} |E'| - |\sigma_e(A')| + \sum_{A' \subseteq E'} (-1)^{|A'|} |E'| - |\sigma(A')|$$

$$= p(\mathcal{K}/e; \lambda) + \lambda p(\mathcal{K} \setminus e; \lambda).$$

Thus (11.34) has been shown.

Finally we shall prove (11.35). From the assumption that $E \setminus e \neq \emptyset$, we have $p(\mathcal{K} \setminus e; 1) = 0$. Hence from the definition (4.9) and the recursion relation (11.34), it is immediate to show

$$\beta(\mathcal{K}) = -\frac{dp(\mathcal{K}/e; \lambda)}{d\lambda}(1) = -\frac{dp(\mathcal{K} \setminus e; \lambda)}{d\lambda}(1) - \frac{dp(\mathcal{K} \setminus e; \lambda)}{d\lambda}(1) = -\beta(\mathcal{K}/e) + \beta(\mathcal{K} \setminus e).$$

This completes the proof. \(\square\)

Contrary to the case of matroids, $\beta(\mathcal{K}) = 0$ does not confirm the separability of a convex geometry. For instance, take three vertices of a triangle in a plane, and insert a new point to two of the three edge. Then the point configuration of these five points on a plane gives a connected convex geometry, while the value of the $\beta$-invariant is null since the absolute value of the $\beta$-invariant of a point configuration is known to be equal to the number of its interior points.

Let $R(\text{rd}; x, y)$ denote the rank generating function derived from a closure rank function $\text{rd}$. Let us denote it by $R(\text{rd}; x, y)$. Then it is easy to get

$$R(\text{rd}; x, y) = \sum_{A \subseteq E} x^{\text{rd}(E) - \text{rd}(A)} y^{|\sigma(A)| - |\text{rd}(A)|} = \sum_{A \subseteq E} x^{|\sigma(A)|} y^{|\sigma(A)| - |A|}$$

(11.36)
For an antimatroid

Theorem 11.20 (Armstrong [9])

An antimatroid $F$ satisfies the direct-sum factorization and a deletion-contraction rule as below.

(1) If $K = K_1 \oplus K_2$, then $R_{cl}(K; x, y) = R_{cl}(K_1; x, y)R_{cl}(K_2; x, y)$.

(2) If $e$ is a coloop of $(E, K)$ and $E \setminus e \neq \emptyset$, then

$$R_{cl}(K; x, y) = xR_{cl}(K/e; x, y) + R_{cl}(K \setminus e; x, y).$$

(Proof) (1) is obvious from Proposition 5.18. The proof of (2) is completely analogous to that of (11.34). Actually, we have

$$R_{cl}(K; x, y) = \sum_{A \subseteq E, e \in A} x^{\sigma(A)}y^{\sigma(A) \setminus |A|} + \sum_{A \subseteq E, e \notin A} x^{\sigma(A)}y^{\sigma(A) \setminus |A|},$$

$$= \sum_{A' \subseteq E \setminus e} x^{\sigma(A')}y^{\sigma(A') \setminus |A'|} + \sum_{A' \subseteq E \setminus e} x^{\sigma(A')}y^{\sigma(A') \setminus |A'|},$$

$$= x \sum_{A' \subseteq E \setminus e} x^{\sigma(A')}y^{\sigma(A') \setminus |A'|} + R_{cl}(K \setminus e; x, y),$$

$$= xR_{cl}(K/e; x, y) + R_{cl}(K \setminus e; x, y).$$

$\square$

11.3 Supersolvable antimatroids

An antimatroid $F$ on $E$ is a supersolvable antimatroid if $F$ is a supersolvable lattice.

Theorem 11.20 (Armstrong [9]) For an antimatroid $F$ on $E$, the following statements are equivalent.

(1) There exists a total order $\preceq_E$ on $E$ satisfying the following:

For feasible sets $A, B \in F$, if $B \subsetneq A$, then $A \cup x \in F$ for the element $x = \min_{\preceq_E}(B \setminus A)$.

(2) The lattice $F$ is an $S_n$ EL-labelling.

(Proof) (1) $\Rightarrow$ (2) For each covering pair $X \prec Y$ in $F$, $\{x\} = Y \setminus X$ for an element $x$ in $E$. We shall show that this is an $S_n$ EL-shelling with respect to the total order $\preceq_E$. By the strong accessibility of $(AC \sqcup)$ of an antimatroid, every maximal chain in an interval $[A, B]$ in $F$ is elementary: $A = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_k = B$ and $\{e_j\} = A_j - A_{j-1}$ for $j = 1, \ldots, k$. The sequence $(e_1, e_2, \ldots, e_k)$ is the labelling of this chain, and a permutation of the elements in $B - A$. The assumption of (1) assures that there exists uniquely an increasing chain in $[A, B]$. Hence this is an $S_n$ EL-labelling.

(2) $\Rightarrow$ (1) Let $\preceq_E$ be a total order on $E$ such that the edge-labelling $f$ derived from $\preceq_E$ is an $S_n$ EL-shelling. Let $A, B \in F$ with $B \subsetneq A$. For the interval $[A, A \cup B]$, by assumption, there is uniquely an increasing chain in it: $A = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_k = A \cup B$. For an element $x$ with $\{x\} = A_1 - A_0$, $x = \min_{\preceq_E}((A \cup B) - A) = \min_{\preceq_E}(B \setminus A)$ holds. So $A_1 = A \cup x \in F$. This completes the proof. $\square$

The equivalency of the supersolvability and the $S_n$ EL-shellability is shown in Theorem 4.28. Summarizing Theorem 4.28 and 11.20, we attain the following theorem.
Theorem 11.21 ([9]) For an antimatroid $\mathcal{F}$ on $E$, the following statements are equivalent.

1. There exists a total order $\leq_E$ on $E$ satisfying the following property:
   
   For feasible sets $A, B \in \mathcal{F}$, if $B \not\subseteq A$, then $A \cup x \in \mathcal{F}$ for the element $x = \min_{E}(B \setminus A)$.

2. The lattice $\mathcal{F}$ is $S_n$ EL-shellable.

3. The lattice $\mathcal{F}$ is supersolvable.

Since an antimatroid is an upper semimodular lattice, Theorem 4.27 immediately implies the following.

Corollary 11.22 The characteristic polynomial of a supersolvable antimatroid factors completely into the product of linear terms.

11.4 Factorizations of characteristic polynomials of 2-spanning convex geometries

The broken circuits of a distributive convex geometry are all of size one, and vice versa. The sizes of broken circuits of poset convex geometries, transitivity convex geometries, and chordal convex geometries are all two.

Let us call such a convex geometry $k$-spanning that the minimal broken circuits are all of size $k$. And we say that a convex geometry is $k$-cornered if the sizes of broken circuits are all $k$. Obviously $k$-cornered implies $k$-spanning.

In a 2-spanning convex geometry, each minimal broken circuit can be considered as an edge between the elements of the underlying sets, from which we can define a graph, a broken circuit graph, with the vertex set being the underlying set and the edge set being the minimal broken circuits. And we call the complement graph of a broken circuit graph an nbc-graph.

Proposition 11.23 An NBC complex of a 2-spanning convex geometry $K$ is equal to the clique complex of the nbc-graph of $K$.

(Proof) The proof is straightforward. □

Corollary 11.24 Let $G$ a connected chordal graph. Then the nbc-graph of a chordal convex geometry associated with $G$ is equal to $G$.

As is described in Theorem 6.20 (3), 2-spanning matroid is necessarily supersolvable, and the corresponding characteristic polynomial factors over nonnegative integers, while the characteristic polynomial of a 2-spanning convex geometry does not necessarily factorize over integers.

For instance, the characteristic polynomial $p_M(\lambda)$ of the graphic matroid of the chordal graph shown in Fig. 11.1 factors over nonnegative integers, while the characteristic polynomial $p_K(\lambda)$ of the chordal convex geometry of the same chordal graph does not factorize. Actually,

$$p_M(\lambda) = (\lambda - 1)^2(\lambda - 2), \quad p_K(\lambda) = \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 1).$$

Let $K$ be a 2-spanning convex geometry, and $G(K)$ be its nbc-graph. Let $\Delta(G(K))$ denote the clique complex of $G(K)$. Then we can deduce from Proposition 11.23 and Theorem 10.9 that
Lemma 11.26

Let $\Delta$ be a pure simplicial complex. $\Delta$ is strongly connected if each pair of facets $F, H \in \Delta^d$ can be connected by a sequence of facets $F = F_0 F_1 \cdots F_t = H$ such that $\dim(F_{i-1} \cap F_i) = d - 1$ for $i = 1, \ldots, t$.

For a $d$-dimensional pure simplicial complex, $(f_d, f_{d-1}, \ldots, f_0)$ is the $f$-vector of $\Delta$, where $f_i$ is the number of $i$-dimensional faces of $\Delta$. The polynomial $f(x) = f_d x^{d+1} + f_0 x^d + \cdots + f_d x^0$ is the $f$-polynomial of $\Delta$. The $h$-polynomial of $\Delta$ is defined as $h(x) = h_0 x^{d+1} + h_1 x^d + \cdots + h_{d+1} x^0 = f(x-1)$, where the coefficients $(h_0, h_1, \ldots, h_{d+1})$ is the $h$-vector of $\Delta$.

A $d$-dimensional pure simplicial complex $\Delta$ is shellable if there is an ordering, a shelling, of its facets $F_1, F_2, \ldots, F_f$ such that $(\bigcup_{i=1}^{j-1} F_i) \cap F_j$ is a pure $(d-1)$-dimensional simplicial complex. It is well-known that $h$-vector is calculated from its shelling $F_1, F_2, \ldots, F_f$, if $\Delta$ is shellable, by

$$h_i = \#\{j : (\bigcup_{k=1}^{j-1} F_k) \cap F_j \text{ has } i \text{ (d-1)-dimensional faces}\}. \quad (11.39)$$

For more details, we refer to [23, 164].

Lemma 11.26 Let $\Delta(G)$ be a clique complex of a chordal graph $G$. Then the following are equivalent.

(i) $\Delta(G)$ is pure and strongly connected,

(ii) $\Delta(G)$ is pure and shellable,

(iii) $\Delta(G)$ is pure and shellable, and there is a shelling $F_1, F_2, \ldots, F_f$ satisfying that $(\bigcup_{i=1}^{j-1} F_i) \cap F_j$ consists of one $(d-1)$-face and its subfaces for each $2 \leq i \leq f$.

(iv) $\Delta(G)$ is pure and shellable, and any shelling $F_1, F_2, \ldots, F_f$ satisfies that $(\bigcup_{i=1}^{j-1} F_i) \cap F_j$ consists of one $(d-1)$-face and its subfaces for each $2 \leq i \leq f$.

(Proof) (iv)⇒(iii)⇒(ii)⇒(i) is straightforward. For (ii)⇒(i), see Björner [23], for example.

To show (i)⇒(iii), we use an induction on the number of facets. When there is only one facet, the statement is obvious. Assume there are $f \geq 2$ facets. Since $G$ is chordal, there is a simplicial vertex $v$ of $G$. There is a unique facet $F$ of $\Delta(G)$ which $v$ belongs to. Since $\Delta(G)$ is pure, $\dim F = \dim \Delta := d$. Let $H$ be the $(d-1)$-dimensional face of $F$ disjoint from $v$. The face $H$ should be incident to a facet other than $F$, otherwise $\Delta$ is not strongly connected. Thus we observe $\Delta(G \setminus v) = \Delta(G) \setminus v = \Delta(G) - \overline{F}$. Now it is easy to verify that $\Delta(G \setminus v)$ is pure and strongly connected with $f-1$ facets, while obviously $G \setminus v$...
is chordal. Thus, by induction hypothesis, $\Delta(G \setminus v)$ is shellable with a shelling $F_1, F_2, \ldots, F_{f-1}$ such that $(\bigcup_{i=1}^{f-1} F_i) \cap F_f$ consists of one $(d-1)$-dimensional face and its subfaces for each $2 \leq j \leq f - 1$. We conclude that $F_1, F_2, \ldots, F_{f-1}, F$ is a shelling of $\Delta(G)$ satisfying the condition of (iii).

To show (iii)$\Rightarrow$(iv), observe that (iii) implies that the $h$-vector of $\Delta(G)$ is $(h_0, h_1, h_2, \ldots, h_{d+1}) = (1, f-1, 0, \ldots, 0)$, where $f$ is the number of facets of $\Delta(G)$, since $F_1$ contributes to $h_0$ and $F_2, \ldots, F_f$ all contribute to $h_1$ in (11.39), for the shelling of the statement. Because the $h$-vector is an invariant of $\Delta$, this implies that all other shellings should satisfy the same condition of (iii).

□

Now we shall state our main theorem.

**Theorem 11.27** Let $K$ be a 2-spanning convex geometry, and $G$ be the nbc-graph of $K$. Then if $G$ is chordal and the clique complex $\Delta(G)$ is pure and strongly connected, then the characteristic polynomial of $K$ factorizes over nonnegative integers as

$$p(\lambda; K) = \lambda^{n-(d+1)}(\lambda - 1)^d(\lambda - f) \tag{11.40}$$

where $n$ is the size of the underlying set, $d$ the dimension of $\Delta(G)$, and $f$ the number of facets of $\Delta(G)$.

(Proof) Using Corollary 11.25 and the equality $f(x) = h(x + 1)$, we have

$$p(\lambda; K) = f_{-1}\lambda^{n-0} - f_0\lambda^{n-1} + f_1\lambda^{n-2} - \cdots + (-1)^{d+1}f_d\lambda^{n-(d+1)}$$

$$= \lambda^{n-(d+1)}(-1)^{d+1}[f_{-1}(-\lambda)^{d+1} + f_0(-\lambda)^d + f_1(-\lambda)^{d-1} - \cdots + (-1)^{d+1}f_d(-\lambda)^0]$$

$$= \lambda^{n-(d+1)}(-1)^{d+1}[h_0(-\lambda + 1)^{d+1} + h_1(-\lambda + 1)^d + \cdots + h_{d+1}(-\lambda + 1)^0].$$

Now because $(h_0, h_1, \ldots, h_{d+1}) = (1, f-1, 0, \ldots, 0)$ from the proof of Lemma 11.26, we have

$$p(\lambda; K) = \lambda^{n-(d+1)}(-1)^{d+1}([-\lambda + 1)^{d+1} + (f - 1)(-\lambda + 1)^d]$$

$$= \lambda^{n-(d+1)}(-1)^{d+1}(-\lambda + 1)^d([-\lambda + 1) + (f - 1)] = \lambda^{n-(d+1)}(\lambda - 1)^d(\lambda - f). \quad \Box$$

Our theorem does not seem to belong to any of the three categories of factorization theorems mentioned in Section 1, and there does not seem to be any similar results so far ever known.

We deduce a special case of Theorem 11.27. From Proposition 11.23, Corollary 11.24, Lemma 11.26 and Theorem 11.27, we have

**Corollary 11.28** Let $G$ be a connected chordal graph. If the clique complex of $G$ is pure and strongly connected, then the characteristic polynomial of the chordal convex geometry associated with $G$ factorizes over nonnegative integers as in (11.40).

We shall display some examples. Let $p_1, p_2, p_3$ be the characteristic polynomial of the chordal convex geometry of the chordal graph $G_1, G_2, G_3$ in Fig. 11.2, respectively. Then

$$p_1(\lambda) = (\lambda - 1)^3, \quad p_2(\lambda) = \lambda(\lambda - 1)^2(\lambda - 2), \quad p_3(\lambda) = \lambda^2(\lambda - 1)^2(\lambda - 3).$$
Figure 11.2: Three pure and shellable chordal graphs
Chapter 12

Topics on convex geometries and antimatroids

12.1 Zeta polynomials of convex geometries

Recall that the zeta polynomial $Z(P, n) = \zeta^n(0, 1)$ of a finite poset $P$ is a polynomial in the variable $n$.

Let $f$ be a function from $E$ to $[n] = \{1, 2, \ldots, n\}$. For a closure system $(K, E)$, $f$ is extremal if for every closed set $X \in K$, \{x \in X : f(x) = \max_{z \in K} f(z)\} \cap \text{ex}(K) \neq \emptyset$. If \{x \in X : f(x) = \max_{z \in K} f(z)\} \subseteq \text{ex}(X)$ for every closed set $X \in K$, $f$ is called strictly extremal.

**Theorem 12.1 (Edelman and Jamison [63])** For $(K, E)$ a convex geometry and $n > 0$,

- $Z(K, n) = \text{the number of extremal functions } f : E \rightarrow [n]$, 
- $(-1)^{|E|} Z(K, n) = \text{the number of strictly extremal functions } f : E \rightarrow [n]$.

In advance Stanley [150, 151] proved for a distributive lattice $L$ the following.

- $Z(K, n) = \text{the number of order-preserving maps } f : L \rightarrow [n]$, 
- $(-1)^{|E|} Z(K, n) = \text{the number of strict order-preserving maps } g : E \rightarrow [n]$.

12.2 Convex dimension of a convex geometry and concave dimension of an antimatroid

It directly follows from Theorem 4.29 and Proposition 4.30 that

**Proposition 12.2 ([67])** For any dual pair $(A, K)$ of an antimatroid and a convex geometry, $\text{cdim}(K) \geq \text{dim}(K)$ and $\text{cadim}(A) \geq \text{dim}(A)$. 
12.3 Caratheodory number, Erdős-Szekeres number and the Helly number

For convex geometries, the Caratheodory number, the Erdős-Szekeres number, and the Helly number are determined from its broken circuits, independent sets, and free sets.

The Helly number of a convex geometry is defined to be the least integer \( m \) for which the following property holds:

\[
\forall H \subseteq \mathcal{K} \quad \text{for subfamily } H' \subseteq H \text{ with } |H'| \leq k \quad \text{if } \bigcap H' \neq \emptyset \quad \text{then } \bigcap H \neq \emptyset
\]

Theorem 12.3 ([93, 95]) The Helly number of a convex geometry is equal to the maximum size of its free sets.

Example 12.4 Theorem 12.3 is not true in general for closure systems. \( \mathcal{K} = \{\emptyset, \{a, b\}, \{c, d\}, \{e, f\}, \{a, b, c, d\}, \{c, d, e, f\}, \{a, b, e, f\}\} \) is a counter-example.

The Caratheodory number of a convex geometry is the least integer \( d \) for which the following holds: for an arbitrary set \( A \subseteq E \) and an element \( a \in A \), there exists a subset \( A' \subseteq A \) such that \( |A'| \leq d \) and \( a \in \sigma(A') \).

Theorem 12.5 ([112]) The Caratheodory number of a convex geometry is equal to the maximum size of its broken circuits.

(Proof) Obvious from the definition. \( \square \)

The Erdős-Szekeres number of a convex geometry is the smallest integer \( k \) such that every family \( \mathbb{H} \subseteq \mathcal{K} \) of closed sets has a subfamily \( \mathbb{H}' \subseteq \mathbb{H} \) with \( |\mathbb{H}'| \leq k \) and \( \cap\{X : X \in \mathbb{H}\} = \cap\{Y : Y \in \mathbb{H}'\} \).

Theorem 12.6 ([112]) The Erdős-Szekeres number of a closure system is equal to the maximum size of its independent sets.

(Proof) Obvious from the definition. \( \square \)

12.4 Caratheodory number of affine convex geometries

凸幾何のサーキットの broken circuit のサイズの最大値を Caratheodory 数と呼ぶ。

有界なセミモジュラー束で

\[ P_{x,y} = \{z \leq y \mid r(x \lor z) - r(z) = r(x \lor y) - r(y)\} \]

とおく。\( x \leq y \) のときは \( P_{x,y} = [x, y] \).

任意の元 \( x, y \in L \) に対して \( P_{x,y} \) が最小限を持つとき、\( L \) は pseudomodular 束であるという。

次のことが知られている。

- Caratheodory 数が 1 であることは必要十分条件は、凸幾何が分配束になることである。
Lemma 12.7 n−次元ユークリッド空間 $\mathbb{R}^n$ 中の d−次元多面体 P、P の頂点集合を E とする多面体への P の单体分割が存在する。

(Proof) d に関する帰納法を用いる。 d=1 のときは明らか。 d ≥ 2 とする。

P の境界 $S = \partial P$ は d−1 次元多面体の連続体で、その各 facet は d−1 多面体になる。ゆえに、帰納法の仮定から、その頂点を台集合とした d−1 単体に单体分割できる。その単体的複体を $\{\Delta_i^{E} : i = 1, \ldots, j_F\}$ とする。P の任意の頂点 v をひとつのとる。このとき、

$$\Delta = \bigcup \Delta_i^{E} : F \text{ is a facet of } S, i = 1, \ldots, j_F \bigcup \bigcup (\Delta_i^{E} \cup \{v\}) : F \text{ is a facet of } S, i = 1, \ldots, j_F$$

は E 上の単体的複体でかつ $\Delta$ の実現は P の単体分割をなしている。ゆえに、P は E を台集合とする d 単体に単体分割できる。 \qed

Proposition 12.8 (E, R) を $\mathbb{R}^n$ 中の disjoint な有限点集合の組で、$R \neq \emptyset$ とする。このとき (E, R) 上の kernel−shelling のサーキットの stem のサイズは n 以下である。

(Proof) $(X, v)$ を任意のサーキットとする。つまり $v \in \text{conv}(X \cup R)$, $X \subseteq E \setminus v$ かつ X はこの性質に関して極小とする。$e \in \text{conv}(X \cup R')$ となる極小な $R' \subseteq R$ をとる。

すると、自明に $R'$ の要素はすべて多面体 $P = \text{conv}(X \cup R')$ の頂点になっている。同様にサーキットの極小性から X の元もすべて P の頂点になりかつ P の頂点集合は X ∪ R' になる。

$R' \neq \emptyset$ として一様性を失わない。もし、$R' = \emptyset$ であるとすると、R が非空であるか、ある元 $w \in R$ をとる。w は P の内点になっている。w で P の各 facet を lifting してできた多面体の全体は P の多面体分割をなす。

この分割の中で v を含む多面体が存在する。それを P' とする。これを言いかえると、P' の頂点集合がある $X' \subseteq X$ に対して $X' \cup w$ なることを意味する。ゆえに、$v \in \text{conv}(X' \cup w) \subseteq \text{conv}(X' \cup R)$ であり、サーキットの極小性から $X' = X$ でなければならない。ゆえに、多面体 P のかわりに $\text{conv}(X \cup w)$ を考えれば、$R' \neq \emptyset$ の場合に帰着できる。

ゆえに、一般性を失わずに $R' \neq \emptyset$ としてよい。R' の任意の元 r をとる。r は P の頂点になっている。

ここで 補題 12.7 の証明より P はどの d−単体も r を含むような d−単体の族に単体分割できる。ゆえに V をふくむある d−単体 $\Delta_r$ がある。

$r$ の $\Delta_r$ 以外の頂点を $X'' \subseteq X$ とすれば、$|X''| = d$ かつ $v \in \text{conv}(X'' \cup r)$. 再びサーキットの極小性から $X'' = X$ でなければならない。ゆえに

$$|X| = |X''| = d \leq n$$

証明終わり。 \qed

Corollary 12.9 Affine 空間中の点配置 E で、$\text{conv.hull}(E)$ が d−次元多面体になるとき、この点配置のアフィン凸幾何の stem のサイズは d + 1 以下である。つまり、Caratheodory 数が d + 1 以下になる。

Corollary 12.10 clutter $\{\{1, 2, 3\}, \{4, 5, 6\}\}$ は、3 次元 affine−kernel shelling の stemu clutter として実現できない。
12.5 Lattices of closure systems and convex geometries

Recall that \( PF(E) \) is the collection of the path independent functions on \( E \), and \( CG(E) \) is the collection of the convex geometries on \( E \).

For choice functions, we introduce a relation \( f \leq g \) if \( f(A) \subseteq g(A) \) for every \( A \subseteq E \). For closure operators, we define \( \sigma \leq \tau \) if \( \tau(A) \subseteq \sigma(A) \) for every \( A \subseteq E \). (これはクロージャーの順序とクロージャーシステムの包含が一致していないことを示します。\( CG(E) \)の意味を凸幾何においてanti-exchangeクロージャーのクラスを\( A\text{Ext}(E) \)として、区別しないとまずい。)

Both \( PF(E) \) and \( CG(E) \) constitute lattices with respect to these partial orders. The restriction of the bijection between \( CF(E) \) and \( Ext(E) \) of Theorem 14.1 naturally gives a bijection between \( PF(E) \) and \( CG(E) \). Actually, this one-to-one correspondence is already shown in Theorem 14.15. \( CG(E) \) is a \( \lor \)-semisublattice of all closure operators (Caspard and Monjardet [39].)

**Theorem 12.11 ([39, 48])** There is a bijection between \( PF(E) \) and \( CG(E) \) which is an antitone isomorphism of lattices.

A union of extreme functions of two convex geometries is again an extreme function of a convex geometry.

**Theorem 12.12 ([29])** Let \( f_1 \) and \( f_2 \) be path independent choice functions. Then \( g = f_1 \cup f_2 \) is a path independent choice function.

(Proof) Let \( g = f_1 \cup f_2 \). First note that \( f_2(A) \subseteq A \) and \( f_1(A) \subseteq A \). Then by the path independency,

\[
g(g(A) \cup B) = g(f_1(A) \cup f_2(A) \cup B) = f_1(f_1(A) \cup f_2(A) \cup B) \cup f_2(f_1(A) \cup f_2(A) \cup B)
\]

\[
= f_1(A \cup f_2(A) \cup B) \cup f_2(f_1(A) \cup A \cup B) = f_1(A \cup B) \cup f_2(A \cup B) = g(A \cup B)
\]

\[\square\]

上の結果がさらに強く次のように言えるか確かめること！

**Theorem 12.13 ([29])** Let \( (\tau_1, \text{ex}_1) \) and \( (\tau_2, \text{ex}_2) \) be pairs of a closure function and an extreme function of convex geometries \( (K_1, E) \) and \( (K_2, E) \), respectively. Then \( \text{ex}(A) = \text{ex}_1(A) \cup \text{ex}_2(A) \) \( (A \subseteq E) \) is an extreme function of a convex geometry \( (K, E) \) such that \( \tau(A) = \tau_1(A) \cap \tau_2(A) \).

(*) 上の \( \tau(A) = \tau_1(A) \cap \tau_2(A) \) がanti-exchangeであることと、\( \text{ex}^* = \tau \)が成立していることを示すこと。

(*) 上の対応が、凸幾何のクロージャーシステムの言葉で \( K_1 \cap K_2 = K_1 \cap K_2 \) なのか \( K_1 \cap K_2 = \{ X \cap Y : X \in K_1, Y \in K_2 \} \) なのか明らかにすること。もし前者であれば、後者のなすmeet-semilatticeはどのような意味を持つことになるのか？

Let \( L \) be the collection of all the closure systems on \( E \). \( L \) is clearly a closure system itself in \( 2^E \). The associated closure operator is

\[
\Phi(K) = \bigcap \{ K' \in L : K \subseteq K' \}.
\]

Since \( L \) has the maximum element \( 2^E \), \( L \) is a lattice. For \( K_1, K_2 \in L \), their union and meet in this lattice are

\[
K_1 \cup K_2 = \Phi(K_1 \cup K_2) = \{ X_1 \cap X_2 : X_1 \in K_1, \ X_2 \in K_2 \},
\]

\[
K_1 \cap K_2 = K_1 \cup K_2.
\]

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Theorem 12.14 ([39]) \( \mathcal{L} \) is a lower locally distributive lattice (or a meet-distributive lattice).

Proposition 12.15 ([33]) Let \((K_i, \sigma_i, E) \ (i \in I)\) be a collection of convex geometries on the same underlying set \(E\). Let us define

\[
\bigwedge_{i \in I} K_i = \{ \bigcap_{i \in I} X_i : X_i \in K_i \ (i \in I) \} \\
\bigwedge_{i \in I} \sigma_i(A) = \bigcap_{i \in I} \sigma_i(A) \quad (A \subseteq E)
\]

Then \(\bigwedge_{i \in I} K_i\) is a closure system on \(E\) whose closure function is \(\bigwedge_{i \in I} \sigma_i\).

(Proof) Obviously, \(\bigwedge_{i \in I} K_i\) is a closure system. First let us show that \(\sigma = \bigwedge_{i \in I} \sigma_i\) is a closure function. It immediately follows from the definition that (i) \(A \subseteq \sigma(A)\) and (ii) \(A \subseteq B\) implies \(\sigma(A) \subseteq \sigma(B)\). Now the idempotency (iii) \(\sigma(\sigma(A)) = \sigma(A)\) is to be proved. By the definition of closure functions, we have

\[
\sigma_i \left( \bigcap_{j \in I} \sigma_j(A) \right) \subseteq \sigma_i(\sigma_i(A)) = \sigma_i(A). \\
(\ i \in I \)
\]

Hence,

\[
\sigma(\sigma(A)) = \bigcap_{i \in I} \sigma_i(\sigma(A)) = \bigcap_{i \in I} \sigma_i \left( \bigcap_{j \in I} \sigma_j(A) \right) \subseteq \bigcap_{i \in I} \sigma_i(\sigma_i(A)) = \bigcap_{i \in I} \sigma_i(A) = \sigma(A)
\]

Thus \(\sigma(\sigma(A)) \subseteq \sigma(A)\). From (i), \(\sigma(\sigma(A)) = \sigma(A)\) follows.

Let \(K_{\sigma} = \{ A \subseteq E : \sigma(A) = A \}\), and lastly we have to show \(K_{\sigma} = \bigwedge_{i \in I} K_i\). By definition, \(K_{\sigma} \subseteq \bigwedge_{i \in I} K_i\) is obvious. We shall prove the converse. It is sufficient to show that for every \(X_i \in K_i \ (i \in I)\), \(\sigma(\bigcap_{i \in I} X_i) = \bigcap_{i \in I} X_i\) holds. Now

\[
\sigma(\bigcap_{j \in I} X_j) = \bigcap_{i \in I} \sigma_i \left( \bigcap_{j \in I} X_j \right) \subseteq \bigcap_{i \in I} \sigma_i(X_i) = \bigcap_{i \in I} X_i
\]

Then it follows from (i) that \(\sigma(\bigcap_{i \in I} X_i) = \bigcap_{i \in I} X_i\). \(\square\)

Theorem 12.16 (Borovik and Gelfand [33]) Let \((K_i, \sigma_i, E) \ (i \in I)\) be a collection of convex geometries. Then \(\bigwedge_{i \in I} K_i, \bigwedge_{i \in I} \sigma_i, E\) is also a convex geometry.

(Proof) It is sufficient to prove that \(\sigma = \bigwedge_{i \in I} \sigma_i\) meets the anti-exchange property. Suppose contrarily that the anti-exchange property is violated. Then there exist \(x, y \in E\) and \(A \subseteq E\) such that \(x \neq y, x, y \notin \sigma(A), x \in \sigma(A \cup y), \) and \(y \in \sigma(A \cup x)\). Since \(x \notin \sigma(A), x \notin \sigma_k(A)\) for some \(k \in I\). By definition, it is trivial that (i) \(x \in \sigma(A \cup y) \subseteq \sigma_k(A \cup y), \) (ii) \(y \in \sigma(A \cup x) \subseteq \sigma_k(A \cup x).\) Now if \(y \notin \sigma_k(A)\), the anti-exchange property leads to \(y \notin \sigma_k(A \cup x)\), contradicting (ii). Hence \(y \in \sigma_k(A)\) must hold. Then \(A \cup y \subseteq \sigma_k(A)\) holds, which signifies \(\sigma_k(A) \subseteq \sigma_k(A \cup y) \subseteq \sigma_k(\sigma_k(A)) = \sigma_k(A).\) Hence we have \(\sigma_k(A \cup y) = \sigma_k(A)\), from which \(x \in \sigma_k(A \cup y) = \sigma_k(A)\) follows, contradicting (i). Thus \(\sigma\) satisfies the anti-exchange property. \(\square\)

Hence the collection \(\mathcal{C}\) of the convex geometries on \(E\) is a closure system in \(2^E\). Since \(\mathcal{C}\) has the maximum element \(2^E, \mathcal{C}\) is a lattice. For \(K_1, K_2 \in \mathcal{C}\), their meet and union in this lattice are

\[
K_1 \lor K_2 = \bigwedge \{ K \in \mathcal{C} : K \cup K_2 \subseteq K \}, \\
K_1 \land K_2 = \bigwedge \{ K \in \mathcal{C} : K \cap K_2 \subseteq K \}
\]

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Chapter 13

Infinite Convex geometries

13.1 Convex geometries of finite characters on infinite sets

We first investigate an infinite closure system and an infinite convex geometry in this monograph.

Let \( X \) be a possibly infinite set. A closure system \( \mathcal{K} \) is a possibly infinite collection of subsets of \( X \) such that

1. \( X \in \mathcal{K} \),
2. for any \( L \subseteq \mathcal{K} \), \( \cap L \in \mathcal{K} \).

As is the same before, the associated closure operator is defined by

\[
\tau(A) = \bigcap \{ B \in \mathcal{K} : A \subseteq B \}. 
\] (13.1)

Even if the underlying set is infinite, the definition of a convex geometry is the same with the finite case. That is, a closure space \( (\mathcal{K}, \tau, X) \) is called a convex geometry if \( \tau \) satisfies the antiexchange property. A closure space \( (\mathcal{K}, \tau, X) \) is of finite character if for any \( A \subseteq E \),

\[
\tau(A) = \bigcup \{ \tau(B) : B \text{ is a finite subset of } A \} 
\] (13.2)

It is equivalent to that the union of any collection of chains of closed sets is again a closed set [41].

Let \( \leq \) be an arbitrary partial order on \( X \). When we define, for \( A \subseteq X \),

\[
\tau_\leq(A) = \{ x \in X : x \leq y \text{ for some } y \in A \} 
\] (y \leq x に書き換える！)

\( \mathcal{I}_\leq = \{ A \subseteq X : \tau_\leq(A) = A \} \) is a poset convex geometry of finite character. A total ordering \( \leq \) of \( X \) is compatible with a closure space \( (\mathcal{K}, \tau) \) if \( \mathcal{I}_\leq \subseteq \mathcal{K} \). The set of total ordering compatible with \( (\mathcal{K}, \tau) \) is denoted \( \text{Comp}(\mathcal{K}) \).

For a given family of closure spaces \( (\mathcal{K}_i, \tau_i, X)_{i \in I} \) on the same ground set, its meet \( (\bigwedge_{i \in I} \mathcal{K}_i, \bigwedge_{i \in I} \tau_i, X) \) is determined by the closure operator

\[
\bigwedge_{i \in I} \tau_i(A) = \bigcap \{ \tau_i(A) : i \in I \}.
\]

In a finite case, the maximal chains of a convex geometry are in one-to-one correspondence to the compatible total orderings under a natural bijection on the ground set. This may fail in infinite cases.
Theorem 13.1 ([154]) Let \((\mathcal{K}, \tau, X)\) be a convex geometry of finite character. Then the correspondence defined below is a one-to-one correspondence between the compatible total orderings for \((\mathcal{K}, \tau, X)\) and the maximal chains in \(\mathcal{K}\).

(Proof) 証明をここに書く。 □

The following theorem holds for closure systems generally.

Theorem 13.2 ([154]) Let \((\mathcal{K}, \tau, X)\) be a closure space. If the correspondence defined by

\[ \preceq \mapsto \mathcal{I}_{\preceq} \]

is a one-to-one correspondence between the compatible total orderings and the maximal chains of \((\mathcal{K}, \tau, X)\), then

\[ \mathcal{K} = \bigwedge_{\preceq \in \text{Comp}(\mathcal{K})} \mathcal{I}_{\preceq} \]

Proposition 13.3 ([154]) If a closure space \((\mathcal{K}, \tau, X)\) satisfies \(\mathcal{K} = \bigwedge_{\preceq \in \text{Comp}(\mathcal{K})} \mathcal{I}_{\preceq}\), then it is a convex geometry.

Theorem 13.4 ([154]) Let \((\mathcal{K}, \tau, X)\) be a closure space of finite character. Then the following are equivalent.

1. \(\preceq \mapsto \mathcal{I}_{\preceq}\) is a one-to-one correspondence between the compatible total orderings and the maximal chains of \((\mathcal{K}, \tau, X)\),
2. \(\mathcal{K} = \bigwedge_{\preceq \in \text{Comp}(\mathcal{K})} \mathcal{I}_{\preceq}\),
3. \((\mathcal{K}, \tau, X)\) is a convex geometry.

If \((\mathcal{K}, \tau, X)\) is a convex geometry of finite character, then (1), (2), and (3) hold. But the converse is not true.

A convex geometry \((\mathcal{K}, \tau, X)\) is disordered if \(A \cup B \notin \mathcal{K}\) for some \(A, B \in \mathcal{K}\). The following is a significant disordered example of a convex geometry.

Example 13.5 For a nonnegative integer \(n > 2\), let us define \(N_n = \{1, 2, \ldots, n\}\) and \(\mathcal{K}_n = \mathcal{P}(\{1, \ldots, n\}) - \{\{1, 2, \ldots, n - 1\}\}. \) It is easy to check that \((\mathcal{K}_n, N_n)\) is disordered.

Proposition 13.6 ([154]) Let \((\mathcal{K}, X)\) be a convex geometry of finite character. Then \((\mathcal{K}, X)\) is disordered if and only if there exists a finite subset \(A \subseteq X\) such that

\[ (\mathcal{K}|A, A) \cong (\mathcal{K}_n, N_n) \] for some \(n > 2\).
13.2 Countable convex geometries and ultrahomogeneity

Recall that a convex geometry \((X, K)\) is atomistic if \(\tau_K(\{e\}) = \{e\}\) for every \(e \in X\).

A convex geometry \((X, K)\) is called countable if the ground set \(X\) is a countable set. A countable convex geometry is ultrahomogeneous if any isomorphism between the substructures of \((X, K)\) can be extended to an automorphism of \((X, K)\). The collection of all the substructures is called the age of \((X, K)\). A general study of ultrahomogeneous structures is found in [91]. The following theorem comes from the results of mathematical logic.

**Theorem 13.7 ([56])** Any two countable ultrahomogeneous convex geometries with the same age are isomorphic.

Let \(N_n\) be a convex geometry on \(\{1, \ldots, n\}\) with the closure system \(K = \{\{i, i + 1, \ldots, j\} : 1 \leq i \leq j \leq n\}\).

**Theorem 13.8 ([56])** Let \((X, K)\) be an ultrahomogeneous convex geometry. If \((X, K)\) contains \(N_3\) as a substructure, then it contains every \(N_n\) for \(n \geq 1\).

For \(a, b \in \mathbb{Q}\) with \(a \leq b\), let \([a, b] = \{c \in \mathbb{Q} : a \leq c \leq b\}\), and \(\mathbb{Q}_{\geq}\) be the collection of finite closed intervals of \(\mathbb{Q}\), i.e. \(\mathbb{Q}_{\geq} = \{[a, b] : a, b \in \mathbb{Q}, a \leq b\}\), while \(\mathbb{Q}_{\leq}\) is a trivial convex geometry on \(\mathbb{Q}\).

A convex geometry is disordered if it is not closed with respect to union.

- \(\mathbb{Q}_{\leq} \times \mathbb{Q}_{\geq}\) denotes a convex geometry on \(\mathbb{Q} \times \mathbb{Q}\) such that the collection \(K_1\) of its closed sets consists of all the elements satisfying
  (1) if \((p, q) \in K_1\), then \((s, t) \in K_1\) for any \(t \in \mathbb{Q}\),
  (2) if \((p, q) \in K_1\), \((p, t) K_1\) and \(q < s < t\), then \((r, s) \in K_1\) for any \(s \in \mathbb{Q}\).

- \(\mathbb{Q}_{\leq} \times \mathbb{Q}_{\geq}\) denotes a convex geometry \(K_2\) on \(\mathbb{Q} \times \mathbb{Q}\) such that
  (1) if \((p, q) \in K_2\) and \(r < q\), then \((p, r) \in K_1\),
  (2) if \((p, q) \in K_2\), \((t, u) K_2\) and \(p < r < t\), then \((r, s) \in K_1\) for any \(s \in \mathbb{Q}\).

Clearly, \(\mathbb{Q}_{\geq}\), \(\mathbb{Q}_{\leq} \times \mathbb{Q}_{\leq}\), and \(\mathbb{Q}_{\leq} \times \mathbb{Q}_{\geq}\) are all countable ultrahomogeneous disordered convex geometry.

**Proposition 13.9 ([56])** A countable atomistic ultrahomogeneous convex geometry of codim 2 is isomorphic to \(\mathbb{Q}_{\geq}\).

**Theorem 13.10 ([56])** Let \((X, K)\) be a countable disordered ultrahomogeneous convex geometry of codim 2. If \((X, K)\) is nonatomistic, then \((X, K)\) is isomorphic to \(\mathbb{Q}_{\geq} \times \mathbb{Q}_{\leq}\) or \(\mathbb{Q}_{\leq} \times \mathbb{Q}_{\geq}\).

**Proposition 13.11 ([56])** There are no atomistic countable ultrahomogeneous disordered convex geometry of codim 3.

13.3 Infinite convex geometries

We shall state the results of [1, 34, 56].
Chapter 14

Intensive and Extensive Operators

The aim of this chapter is first to present the one-to-one correspondence between intensive operators (choice functions) and extensive operators (expanding functions). Furthermore, the heredity intensive operators and the monotone extensive operators are also in one-to-one correspondence. Danilov and Koshevoy [48] established a wide framework including the above one-to-one correspondence introducing the concept of neighbourhoods.

Koshevoy [113] showed that a path-independent choice function is an extreme-point function of a convex geometry, and vice versa. Hence a path independent choice function is cryptomorphic to a convex geometry.

14.1 Intensive operators and extensive operators

We first prepare some definitions. An operator $f$ on a non-empty finite set $E$ is a mapping $f : 2^E \to 2^E$.

1. An operator $f$ is extensive if $A \subseteq f(A)$ for any $A \subseteq E$.
2. An operator $f$ is an intensive operator or a choice function if $f(A) \subseteq A$ for any $A \subseteq E$.
3. An operator $f$ is monotone if $A \subseteq B$ implies $f(A) \subseteq f(B)$.
4. An operator $f$ is idempotent if $f(f(A)) = f(A)$ for any $A \subseteq E$.

Let $\text{CF}(E)$ and $\text{Ext}(E)$ be the collections of the intensive operators and the extensive operators on $E$, respectively.

There is a bijection between $\text{Ext}(E)$ and $\text{CF}(E)$. For an extensive operator $e$, we can define an intensive operator $c = e^+$ by

$$e^+(A) = \{a \in A : a \not\in e(A-a)\}$$  \hspace{1cm} (14.1)

Conversely, for an intensive operator $c$,

$$c^*(A) = A \cup \{x \in E \setminus A : x \not\in c(A \cup x)\}$$  \hspace{1cm} (14.2)

defines an extensive function $\mu = c^*$. 

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The set \( \text{CF}(E) \) of all the intensive operators on \( E \) is a Boolean lattice with respect to the partial order defined by \( f \leq g \) if \( f(A) \subseteq g(A) \) for any \( A \subseteq E \). The maximum element of \( \text{CF}(E) \) is the identity map and the minimum element is a map which attains \( \emptyset \) for every set. In the same way, \( \text{Ext}(E) \) is a Boolean lattice.

**Theorem 14.1 ([49])** For any extensive function \( e \in \text{Ext}(E) \), \((e^+)^* = e\) holds, and \((c^*)^+ = c\) holds and for any intensive operator \( c \). Namely, + and * compose an antitone bijection between \( \text{Ext}(E) \) and \( \text{CF}(X) \).

(Proof) For an extensive function \( e \), let \( c = e^+ \). Then for every \( A \subseteq E \),

\[
(e^+)^*(A) = A \cup \{ x \in E \mid A : x \not\in c(A \cup x) \} = A \cup \{ x \in E \mid x \not\in e^+(A \cup x) \}
\]

\[
= A \cup \{ x \in E \mid A : x \not\in \{ a \in A \cup x : a \not\in e((A \cup x) - a) \} \}
\]

\[
= A \cup \{ x \in E \mid A : x \not\in A \cup x or x \not\in (e((A \cup x) - x)) = e(A)
\]

Similarly, let \( c \) be an intensive operator, and \( e = c^* \). Then for every \( A \subseteq E \),

\[
(c^*)^+(A) = \{ a \in A : a \not\in e(A - a) \} = \{ a \in A : a \not\in c^+(A - a) \}
\]

\[
= \{ a \in A : a \not\in (A - a) \cup \{ x \in (E \mid A) \cup a : x \not\in c(A \cup x) \} \}
\]

\[
= \{ a \in A : a \not\in A - a and a \not\in cA = c(A)\} = c(A).
\]

Suppose \( e_1, e_2 \in \text{Ext}(E) \) and \( e_1 \leq e_2 \). Then for any \( A \subseteq E \), \( e_1^*(A) = \{ a \in A : a \not\in e_1(A - a) \} \) and \( e_2^*(A) = \{ a \in A : a \not\in e_2(A - a) \} \). By assumption, \( e_1(A - a) \subseteq e_2(A - a) \), which implies \( e_1^*(A) \supseteq e_2^*(A) \). Hence \( e_1^* \geq e_2^* \). Thus the bijection is an antitone isomorphism. \( \square \)

### 14.2 Heritage operators and monotone operators

An intensive operator \( f \) on \( E \) has the **heredity property** if for any \( A \subseteq B \subseteq E \),

\[
\text{(H)} \quad A \cap f(B) \subseteq f(A).
\]

The heredity property is equivalent to the following.

\[
\text{(H)} \quad \text{If } a \in f(A) \text{ and } a \in B \subseteq A, \text{ then } a \in f(B).
\]

An intensive operator \( f \) on \( E \) has the **outcast property** (or the Chernoff property) if for \( A, B \subseteq E \),

\[
\text{(O)} \quad \text{If } f(B) \subseteq A \subseteq B, \text{ then } f(A) = f(B).
\]

The outcast property is equivalent to below.

\[
\text{(O)} \quad \text{If } a \not\in f(A), \text{ then } f(A \setminus a) = f(A).
\]

**Proposition 14.2** An intensive operator with the heredity property is idempotent, and an intensive operator with the outcast property is idempotent.

(Proof) Let \( f \) be an intensive operator. For any \( X \subseteq E \), let \( A = f(X) \subseteq X \). Then if \( f \) has the heredity property, \( f(X) = f(X) \cap f(X) = A \cap f(X) \subseteq f(A) = f(f(X)) \). Hence \( f(X) \subseteq f(f(X)) \). Since \( f \) is an intensive operator, \( f(X) = f(f(X)) \) holds.

Suppose \( f \) has the outcast property. By definition, \( f(X) \subseteq A \subseteq X \). The outcast property implies \( f(A) = f(X) = f(f(X)) \). This completes the proof. \( \square \)
Corollary 14.3 If a monotone intensive operator has either the heredity property or the outcast property, it is a closure operator.

Let $\text{Mon}(E)$ be the collection of the monotone extensive operators, and $\text{Her}(E)$ the collection of intensive operators satisfying the heredity property.

Proposition 14.4 ([49]) An intensive operator $c$ has the heredity property if and only if $e = c^*$ is a monotone extensive function. This gives an antitone isomorphism between $\text{Her}(E)$ and $\text{Mon}(E)$.

(Proof) Suppose $c$ to have the heredity property intensive operator, and $A \subseteq B$. We shall show $c^*(A) \subseteq c^*(B)$. Suppose contrarily that there exists an element $x \in c^*(A) \setminus c^*(B)$. $x \in c^*(A)$ implies $x \in A$ or $x \notin c(A \cup x)$. The first case leads to $x \in A \subseteq B \subseteq c^*(B)$, a contradiction. So we can assume $x \notin c(A \cup x)$. By the heredity property, $(A \cup x) \cap c(B \cup x) \subseteq c(A \cup x)$, a contradiction. Hence we have $x \notin c(B \cup x)$, and $x \in c^*(B)$, a contradiction.

Conversely, suppose $e = c^*$ is a monotone extensive function, and $A \subseteq B$. We should show $A \cap c(B) \subseteq c(A)$. Take an arbitrary element $x \in A \cap c(B)$. Since $x \in c(B)$, we have $x \notin c^*(B - x)$. From the monotonicity, $c^*(A - x) \subseteq c^*(B - x) \notin x$ follows, which implies $x \notin c^*(A - x)$ and $x \in c(A)$. This completes the proof. □

The set $\text{Her}(E)$ of the heredity property intensive operator is a distributive sublattice of $\text{CF}(E)$, and $\text{Mon}(E)$ is a distributive sublattice of $\text{Ext}(E)$ [49].

EXERCISE 14.5 Characterize the join-irreducible elements of $\text{Her}(E)$ and $\text{Mon}(E)$.(自分で解くこと！)

section Extreme-point operators and closure operators Let us examine which intensive operators are the extreme-point operators of closure operators. An extreme-point operator can be defined from the closure operator in the following way as well.

Proposition 14.6 ([6]) For a closure space $(K, \sigma, E)$, the extreme-point operator is given by below.

$$\text{ex}(A) = \{ a \in A : a \notin e(A - a) \} = \bigcap \{ B : \sigma(B) = \sigma(A) \}$$

$$\sigma(A) = A \cup \{ x \in E \setminus A : x \notin c(A \cup x) \} = \bigcup \{ B : \text{ex}(B) = \text{ex}(A) : \}$$

(Proof) □

The extreme-point operator of a closure system has the outcast property. That is,
Proposition 14.7 ([6, 127]) For a closure space \((\mathcal{K}, \sigma, E)\), the extreme-point operator \(\text{ex}\) satisfies that \(\text{ex}(B) \cap A \subseteq \text{ex}(A)\) for \(A \subseteq B \subseteq E\).

(Proof) □

Theorem 14.8 ([7]) An intensive operator \(f\) on a non-empty finite set \(E\) is the extreme-point operator of a closure system if and only if it satisfies the following.

(1) If \(A \subseteq B \subseteq E\), then \(f(B) \cap A \subseteq f(A)\). (Chernoff property)

(2) For any \(A \subseteq E\) and \(x, y \in E\) with \(x, y \notin A\), if \(x \in f(A \cup \{x\})\) and \(y \notin f(A \cup \{y\})\), then \(x \in f(A \cup \{x\} \cup \{y\})\).

(Proof) □

14.3 Path-independent operators and convex geometries

The theory of choice functions are so far studied in mathematical social science. Seeking for well-behaved choice functions, they conceived a rationalizable function. The concept of path-independent choice function was introduced by Plott [140] as a weakening of rationality. A path-independent function is also called a Plott function. For the details of Plott functions, we refer to Danilov and Koshevoy [47].

An operator \(f\) is path-independent if

\[
f(A \cup B) = f(f(A) \cup B) \quad \text{for any } A, B \subseteq E. \tag{14.3}
\]

(14.3) is equivalent to

\[
f(A \cup B) = f(f(A) \cup f(B)) \quad \text{for any } A, B \subseteq E. \tag{14.4}
\]

Actually, (14.4) instantly follows from (14.3). Note that a path-independent function satisfies \(f(f(A)) = A \quad (A \subseteq E)\). Conversely, suppose (14.4). Then \(f(A \cup B) = f(f(A) \cup f(B)) = f(f(f(A)) \cup f(B)) = f(f(A) \cup B)\).

The following characterization of path-independent intensive operator is well known in the theory of choice functions.

Theorem 14.9 (Aizerman and Malishevski [4], [47]) An intensive operator \(f\) is path-independent if and only if it is an intensive operator satisfying both the heredity property and the outcast property.

(Proof) Suppose \(f\) is a path-independent intensive operator. First the heredity property is to be shown. Let \(a \in B \subseteq A\) and \(C = A - B\). Then if \(a \in f(A)\), then \(a \in f(B \cup C) = f(f(B) \cup C)\). Since \(f\) is an intensive operator and \(a \notin C\), \(a \in f(B)\). Next the outcast property is to prove. Suppose \(f(A) \subseteq B \subseteq A\). Since \(A = A \cup B\) and \(f(A) \subseteq B\), \(f(A) = f(A \cup B) = f(f(A) \cup B) = f(B)\).

Conversely, let \(f\) be an operator satisfying the heredity property and the outcast property. First we shall show

\[
f(A \cup B) \subseteq f(A) \cup B \quad \text{for any } A, B \subseteq E. \tag{14.5}
\]
Suppose \( e \in f(A \cup B) \). Either \( e \in A \) or \( e \in B \) holds since \( f \) is intensive. If \( e \in \mathcal{B} \), then \( e \in A \) \( f(A \cup B) \) from the assumption, while \( A \cap f(A \cup B) \subseteq f(A) \) follows from the heritage property. Hence \( e \in f(A) \). Thus \( f(A \cup B) \subseteq f(A) \cup B \) is shown.

Setting \( C = f(A) \cup B \), it follows from (14.5) that

\[
f(A \cup B) \subseteq C \subseteq A \cup B.
\]

The outcast property gives instantly \( f(A \cup B) = f(C) = f(f(A) \cup B) \). Hence \( f \) is proved to be path-independent. \( \square \)

From a contracting operator \( c \), as in (14.2), an extensive function is defined by

\[
c^{*}(A) = A \cup \{ x \in E \setminus A : x \notin c(A \cup x) \} \quad (14.6)
\]

By Theorem 14.9, if \( c \) is a path-independent contracting function, \( c \) is hereditary, and \( c^{*} \) is monotone from Proposition 14.4. We can further show that \( c^{*} \) is idempotent.

\[
C^{*}(A) = \bigcup \{ B \subseteq E : C(B) = C(A) \} \quad (14.7)
\]

is derived.

**Proposition 14.10** Let \( \sigma \in \text{Ext}(E) \) be a closure operator on \( E \). If the inclusive function \( C = \sigma^{\#} \) has the path-independent property, then \( C(\sigma(A)) = C(A) \) for any \( A \subseteq E \).

(Proof) \( \Box \)

**Theorem 14.11 ([113])** Let \( \sigma \in \text{Ext}(E) \) be a closure operator on \( E \), and \( \text{ex} \) be the corresponding extreme-point function. Then \( \text{ex} \) has the path-independent property if and only if \( \sigma \) satisfies the anti-exchange property.

**Proposition 14.12 ([113])** For a path independent (hereditary) choice function \( C : 2^{E} \to 2^{E} \), \( C^{*} \) is a closure operator and \( C(C^{*}(A)) = C(A) \).

(Proof) \( A \subseteq C^{*}(A) \) is obvious. We shall prove the idempotency of \( C^{*} \). For \( A \subseteq E \), set \( E_{A} = \{ A' : C(A') = C(A) \} = \{ A_{1}, \ldots, A_{k} \} \). Then \( C^{*}(A) = \bigcup_{i=1}^{k} A_{i} \), and \( C^{*}(A_{i}) = \bigcup \{ A^{*} : C(A^{*}) = C(A_{i}) = C(A) \} \) for \( i = 1, \ldots, k \), so that \( C^{*}(A_{i}) = C^{*}(A) \). We shall prove \( C(C^{*}(A)) = C(\bigcup_{i=1}^{k} A_{i}) = C(A) \) by induction on \( k \). When \( k = 1 \), \( C(C^{*}(A)) = C(A_{1}) = C(A) \). The induction step comes from

\[
C(C^{*}(A)) = C(\bigcup_{i=1}^{j-1} A_{i} \cup (A_{j})) = C(C(\bigcup_{i=1}^{j-1} A_{i} \cup C(A_{j}))
\]

\[= C(C(A) \cup C(A_{j})) = C(C(A)) = C(A).\]
Hence $C(C^*(A)) = C\left(\bigcup_{i=1}^{k} A_i\right) = C(C(A)) = C(A)$. Now we have $C^*(C(C^*(A))) = C^*(C(A))$, and $C^*(C^*(A)) = C^*(A)$ follows from Lemma 14.13.

Lastly we have to show the monotonicity. Suppose $A \subseteq B$, and $C^*(A) \not\subseteq C^*(B)$. Then there exists an element $x \in C^*(A) \setminus C^*(B)$. By definition, $C(X) = C(A)$ for some $X \subseteq S$ with $x \in X$.

If $C(Y) = C(B)$ for $Y$, then $x \notin Y$ follows from the definition. By path independency, $C(B) = C(A \cup B) = C(C(A) \cup B) = C(X \cup B)$. Hence $Y' = X \cup B$ satisfies $C(Y') = C(B)$ and $x \in Y'$, a contradiction. 

Now present an important lemma corresponding to Krein-Milman property.

**Lemma 14.13** Let $C : 2^E \to 2^E$ be a path independent choice function. Then $C^*(C(A)) = C^*(A)$.

(Proof) Since $C(C(A)) = C(A)$, we have

$$C^*(C(A)) = \bigcup\{A' : C(A') = C(C(A))\} = \bigcup\{A' : C(A') = C(A)\} = C^*(A) \tag{14.8}$$

For a closure operator $C^*$, recall that the extreme function is $ex_{C^*}(A) = \{a \in A : a \notin C^*(A - a)\}$. (See (5.2).)

**Lemma 14.14** Let $C : 2^E \to 2^E$ be a path independent choice function, and $C^*$ be the associated closure operator. Then $C(A) = ex_{C^*}(A)$.

(Proof) Suppose $a \in A$ and $a \in C^*(A - a)$. We shall show $a \notin C(A)$. From Proposition 14.12 and the definition of path independency, we have

$$C(A) = C(C(A - a) \cup C(a)) = C(C^*(A - a) \cup C(a)) = C(C^*(A - a) \cup \{a\})$$

$$= C(C^*(A - a)) = C(A - a) \notin a$$

Conversely, suppose $a \in C(A)$. Take $B \subseteq C(A) \setminus a$ such that $A - a = C(A) \cup B$. Then

$$a \in A \subseteq C^*(A) = C^*(C(A)) \subseteq C^*(C(A) \cup B) = C^*(A - a).$$

Hence $a \in ex(A)$. This completes the proof.

**Theorem 14.15** ([113]) Let $C : 2^E \to 2^E$ be a path independent choice function. Then $C^*$ is a closure operator of a convex geometry on $E$, and $C$ is its extreme-point function, that is, $C(A) = ex_{C^*}(A)$.

(Proof) Lemma 14.13 readily implies that $(C^*, C)$ meets the Krein-Milman property of Theorem 7.7 (D), which shows that $C^*$ is a closure operator of a convex geometry.

(Otherwise, $C^*$ can be deduced to enjoy the condition of Theorem 7.7 (E) from Proposition 14.12, and so $C^*$ is proved to be a closure operator of a convex geometry.)

Conversely, a convex geometry gives a choice function satisfying path-independence.

**Theorem 14.16** ([113]) Let $\tau : 2^E \to 2^E$ be a closure operator with anti-exchange property, and $ex(A) = \{e \in A : e \notin \tau(A - e)\}$ be its extreme function. Then $ex$ meets the path independence property.

$$ex(A \cup B) = ex(ex(A) \cup B)$$

(14.8)
In particular, if the closure system derived from $C^*$ is the collection of ideals of a poset $P = (E, \leq)$, they say $C$ is ordinarily rationalizable. The path independency is considered as a weakening of rationalizability.

14.4 Neighborhood systems

Danilov and Koshevoy [49] devised a tight linkage of neighbourhood systems, contracting and extensive operators. This has already appeared in Choice Theory as hyper-relations and the extended partial orders.

As is seen in Proposition 14.4, heritage contracting functions and monotonic extensive functions are cryptomorphic to each other. We can introduce a topological notion ‘neighborhood system’ which is cryptomorphic to these two notions.

A neighborhood system on $E$ is a family $N = (N_x, x \in E)$ with $N_x \subseteq 2^E$ satisfying the two conditions below.

(N1) $x \in U$ for every $U \in N_x$,
(N2) if $U \subseteq N_x$ and $U \subseteq V$, then $V \in N_x$.

When $U \in N_x$, we say $U$ is a neighborhood of the element $x$.

A element $x \in E$ is adherent to a set $A \subseteq E$ if every neighborhood of $x$ intersects $A$, or equivalently if $A^c = E - A$ is not a neighborhood of $x$. If $N_x = \emptyset$, $x$ is said to be dummy. A dummy element is adherent to every subset of $E$. $Adh_N(A)$ denotes the set of all the elements adherent to $A$. Since $Adh_N$ is monotonic and extensive, we have an antitone mapping

$$Adh : NS(E) \to Mon(E).$$ (14.9)

An element $a \in A$ is isolated in $A$ if some neighborhood of $a$ does not intersects with $A - a$. Let $Isi_N(A)$ be the set of all the isolated elements of $A$. For $A \subseteq B$, if $a \in A$ is isolated in $A$, obviously $a$ is isolated in $B$. Hence $Iso_N$ is a heritage contracting function, and we have a monotonic mapping

$$Iso : NS(E) \to Her(E)$$ (14.10)

**Theorem 14.17 ([49])** The mappings $Adh$ and $Iso$ are bijections, and the following diagram is commutative.
Now we shall investigate various types of neighborhood systems adding some conditions.

**Exchange operators**

We shall consider an extensive function which has the exchange property.

For a monotonic extensive operator \( \mu \), the following is the exchange property.

\[(\text{Exc}) \quad \text{Suppose } x, y \notin \mu(A) \text{ and } x \in \mu(A \cup y). \text{ Then } y \in \mu(A \cup x).\]

The counter-part for a contracting function \( f \) is the following condition.

\[(\text{M}) \quad \text{If } x \in f(A \cup x), \text{ then } f(A) \subseteq f(A \cup x).\]

For a neighborhood system \( \mathcal{N} = (\mathcal{N}_x, x \in E) \), we shall consider the following symmetry axiom.

\[(\text{S}) \quad \text{Suppose } U \in \mathcal{N}_x \cap \mathcal{N}_y. \text{ If } U - x \text{ is a neighborhood of } y, \text{ then } U - y \text{ is a neighborhood of } x.\]

In this case the diagram Fig. 14.1 turns out to be the following.

**Proposition 14.18** Let \( \mathcal{N} \) be a neighborhood system, \( \mu = \text{Adh}(\mathcal{N}) \), and \( f = \text{Iso}(\mathcal{N}) \). Then the following are equivalent.

1. The monotonic extensive operator \( \mu \) satisfies the exchange property (Exc),
2. The heritage contracting operator \( f \) satisfies (M),
3. \( \mathcal{N} \) satisfies the symmetry axiom (S).

**Anti-exchange property**

A monotonic extensive operator \( \mu \) satisfies the *anti-exchange axiom* if

\[(\text{AExc}) \quad \text{Suppose } a, b \notin \mu(A). \text{ If } a \in \mu(A \cup b) \text{ and } b \in \mu(A \cup a), \text{ then } a = b.\]

The counter-part condition for a contracting function \( f \) is the following.

\[(\text{N}) \quad \text{Suppose } a \in f(A \cup b) \text{ and } b \in f(A \cup a). \text{ Then } a, b \in f(A \cup a \cup b).\]

A neighborhood system \( \mathcal{N} \) is a *Kolmogorov neighborhood system* if it satisfies

\[(\text{K}) \quad \text{Suppose } U \text{ is a common neighborhood system of two distinct elements } x \text{ and } y. \text{ Then either } U - x \text{ is a neighborhood of } y \text{ or } U - y \text{ is a neighborhood of } x.\]

The Kolmogorov system is weaker than the separation axiom of ordinary topologies.

**Proposition 14.19** ([49]) Let \( \mathcal{N} \) be a neighborhood system, \( \mu = \text{Adh}(\mathcal{N}) \), and \( f = \text{Iso}(\mathcal{N}) \). The following are equivalent.
(1) $N$ is a Kolmogorov neighborhood system,

(2) The extensive operator $\mu$ satisfies the anti-exchange axiom (AExc),

(3) The contracting function $f$ satisfies (N).

**Closure operators and pre-topology**

The most familiar extensive operators are closure operators. Closure operators can be characterized by the following transitivity property.

**Proposition 14.20 ([49])** An extensive operator $\mu$ is a closure operator if and only if it satisfies the property (T) below.

(T) If $b \in \mu(A)$, then $\mu(A \cup b) \subseteq \mu(A)$.

Let us denote by $\text{Clo}(E)$ the poset of closure operators on $E$. $\text{Clo}(E)$ is closed under intersection. Hence for any operator $p$, there exists the minimum closure operator $\mu$ such that $p \leq \mu$. Such $\mu$ is denoted by $\text{cl}(p)$, and called the closure of $p$. $\text{Clo}(E)$ is a meet-semilattice and has the maximum element. Hence it is a lattice with $\wedge$ equal to $\cap$ and with the union $\vee$ given by $\mu_1 \vee \mu_2 = \text{cl}(\mu_1 \cup \mu_2)$.

Suppose a neighborhood $N$ is given. For a set $A$, an element $a \in A$ is an interior element of $A$ if $A$ is a neighborhood of $a$. Let $\text{int}(A)$ be the set of the interior elements of $A$. A set $A$ is called open if $\text{int}(A) = A$.

We shall consider the following condition for neighborhood systems.

(N4) If $U \in N_x$, then $\text{int}(U) \in N_x$.

A heritage function is called closed if it satisfies

(W) If $a \in f(A \cup a)$ and $b \not\in f(A \cup b)$, then $a \in f(A \cup a \cup b)$.

**Proposition 14.21 ([49])** Let $N$ be a neighborhood system, $\mu = \text{Adh}(N)$, and $f = \text{Iso}(N)$. The following are equivalent.

(1) $N$ satisfies (N4),

(2) The extensive operator $\mu$ is a closure operator,

(3) The heritage contracting function $f$ is closed.

**Pre-topology**

A family $O$ of subsets of $E$ is a pre-topology if it is closed under union. An element of $O$ is called an open set. An emptyset is necessarily open. Clearly a pre-topology and a closure system is equivalent by taking the complement.

A pre-topology is cryptomorphic to a neighborhood system satisfying (N4) by Proposition 14.21. Actually, a neighborhood system $N$ determines a pre-topology $\mathcal{N}T(N)$ which consists of all the open sets with respect to $N$. Conversely a pre-topology $P$ gives rise to a neighborhood system $\mathcal{N}S(P)$ which satisfies (N4). Moreover, $\mathcal{N}S(\mathcal{N}T(N)) = N$ if and only if $N$ satisfies (N4).
Now we shall construct a closed choice function \( f = Iso_P \) from a pre-topology by
\[
f(A) = \{ a \in A \mid A \cap U \text{ for some } U \in \mathcal{P} \}.
\] (14.11)

Conversely, a heritage choice function \( h \) gives rise to a pre-topology \( \mathcal{P}(h) \). Let us call a set \( U \) \( h \)-open if \( x \in h((E \setminus U) \cup x) \) for any \( x \in U \). The set \( \mathcal{P}(h) \) of the \( h \)-open sets forms a pre-topology.

In the framework of Theorem 14.1, by Proposition 14.21, there exists uniquely a closure operator \( \mu_h \) corresponding to a heritage choice function \( h \). Furthermore, the diagram Fig. 14.1 is commutative there, and \( \mu_h = cl(h) = Iso_P \) holds.

Hence we have a natural bijection between the lattice \( PreTop(E) \) of pre-topologies and the lattice \( Cl(0) \) of closure operators (or closure systems).

**EXERCISE 14.22** Show that \( \mathcal{P}(h) \) is really a pre-topology for a heritage choice function \( h \).

**The inverse of neighborhood system**

Let \( \mathcal{N} = (N_x, x \in E) \) be a neighborhood system on \( E \). A subset \( U \subseteq E \) is an inverse neighborhood of an element \( x \) if \( x \in U \) and \( U - x \) intersects every neighborhood of \( x \). The collection \( N_x^\circ \) of inverse neighborhoods of \( x \) clearly satisfies (N1) and (N2). Hence \( \mathcal{N}^\circ = (N_x^\circ, x \in E) \) is a neighborhood system, which is called the inverse of \( \mathcal{N} \).

**Proposition 14.23** The inversion \( \mathcal{N} \mapsto \mathcal{N}^\circ \) is an antitone involution of the set \( NS \).

**Proposition 14.24** The inversion swaps the axioms (W) and (M).

**Corollary 14.25** A neighborhood system \( \mathcal{N} \) satisfies (S) if and only if the inverse neighborhood system \( \mathcal{N}^\circ \) is a pre-topology.

**Exchange closure operators**

マトロイドの場合。

We shall consider the closure operators satisfying the exchange axiom (Exc), which are cryptomorphic to matroids. The contracting functions corresponding to matroids are called matroidal. In other words, a contracting function is matroidal if it satisfies (H), (W), and (M).

A matroid can be stated in terms of pre-topologies as follows. A pre-topologies is symmetric if it satisfies

\[(S') \text{ Let } U \text{ be an open set containing elements } x \text{ and } y. \text{ If } U - y \text{ is a neighborhood of } x, \text{ then } U - x \text{ is a neighborhood of } y.\]

**Proposition 14.26** A pre-topology \( \mathcal{P} \) is symmetric if and only if the corresponding neighborhood system \( \mathcal{N} = N_S(\mathcal{P}) \) satisfies (S).

(Proof) \( \square \)

**Proposition 14.27** If a contracting function \( f \) corresponds to a matroid, then the inverse function \( f^\circ \) corresponds to the dual matroid.
Kolmogorov pre-topologies and path-independent functions

A convex geometry is a closure system whose closure operator satisfies the anti-exchange axiom (AExt). The corresponding contracting functions satisfy (H), (W), and (N), and the corresponding pre-topologies are Kolmogorov pre-topologies, i.e. they satisfy (K) and (N4).

The condition set of (H), (W) and (N) is equivalent to that of (H) and (O) or to that of (H) and (O').

\[(O) \text{ If } x \notin f(A) \text{ then } f(A - x) \subseteq f(A),\]
\[(O') \text{ If } f(B) \subseteq A \subseteq B, \text{ then } f(A) = f(B)\]

**Proposition 14.28** A contracting function is path-independent if and only if it satisfies (H) and (O).

(Proof) 自分で考える。 □

**Lemma 14.29** The axiom set of (H) and (O) is equivalent to the one of (H), (N) and (W).

**Corollary 14.30** A contracting function is path-independent if and only if it satisfies (H), (N) and (W).

The following notions are cryptomorphic.

1. path-independent choice functions,
2. anti-exchange closure operators,
3. convex geometries,
Chapter 15

Optimizations and Packing Theorems on Convex Geometries

15.1 The dual greedy algorithm on independent sets of convex geometries

In this section, a linear optimization problem over the independent sets of a convex geometry is investigated. Faigle and Kern [69, 70] and Krüger [115] considered the independent sets of poset convex geometries. Plainly speaking, an independent set of a poset convex geometry is an antichain. Krüger [115] investigated a submodularity-type property on which the dual greedy algorithm of Faigle and Kern [69] works.

The linear optimization considered here is as follows. Let \((K, \tau, E)\) be a convex geometry and \(\text{In} = \text{In}(K)\) be the set of the independent sets of \(K\). \(c \in \mathbb{R}^E_+\) is a nonnegative vector. Suppose \(f: \text{In}(K) \to \mathbb{R}\) to be a function with \(f(\emptyset) = 0\). The primal linear optimization problem to be considered is

\[
\text{Maximize } \sum_{e \in E} c(e)x(e)
\]

subject to

\[
\sum_{e \in E} x(e) \leq f(X) \quad (X \in \text{In}).
\]

The dual problem \((D)\) is

\[
\text{Minimize } \sum_{X \in \text{In}} y(X)f(X)
\]

subject to

\[
\sum_{X \in \text{In}, e \in X} y(X) = c(e) \quad (e \in E),
\]

\[
y(X) \geq 0 \quad (X \in \text{In}).
\]

To solve the above dual problem, Faigle and Kern [69] developed a greedy algorithm \((G)\) shown in Fig. 15.1.

Recall that in the lattice \(K\), \(A \wedge B = A \cap B\) and \(A \vee B = \tau(A \cup B)\) for \(A, B \in K\). We define here a partial order on \(\text{In}(K)\) by \(X \preceq Y\) if \(\tau(X) \subseteq \tau(Y)\) for \(X, Y \in \text{In}(K)\). Since \(\text{ex}: K \to \text{In}(K)\) is a bijection, \(\text{In}(K)\) forms a lattice with respect to this partial order. In this lattice, \(X \wedge Y = \text{ex}(\tau(X) \cap \tau(Y))\) and \(X \vee Y = \text{ex}(\tau(X \cup Y))\) for \(X, Y \in \text{In}(K)\).
Algorithm  Faigle–Kern’s dual greedy algorithm (G)

Input  \( c \in \mathbb{R}^E_+ \)

Output  \( y \in \mathbb{R}^{\text{ex}(L)} \) (an optimum of (D)) and \( \pi \) (a permutation of \( E \))

1: **Initialization:**
2: \( y(X) \leftarrow 0 \) \( (\forall X \in \text{ex}(L)) \);
3: \( T \leftarrow E \);
4: \( w(e) \leftarrow c(e) \) \( (\forall e \in E) \);
5: \( \pi \leftarrow \emptyset \);
6: **Iteration:**
7: \( \text{while } T \neq \emptyset \text{ do} \)
8:  determine \( e \in \text{ex}(T) \) s.t. \( w(e) = \min \{ w(e') : e' \in \text{ex}(T) \} \);
9:  \( y(\text{ex}(T)) \leftarrow w(e) \);
10:  \( \pi \leftarrow e\pi \);
11:  \( w(x) \leftarrow w(x) - w(e) \) \( (\forall x \in \text{ex}(T)) \);
12:  \( T \leftarrow T \setminus \{ e \} \).
13:  end of while

Figure 15.1: Faigle–Kern’s dual greedy algorithm.

We prepare some notations. For \( X, Y \in \text{In}(\mathcal{K}) \),

\[
X \sqcup Y = (X \land Y) \cap (X \lor Y), \quad X \bowtie Y = (X \cap Y) \setminus (X \lor Y)
\]

Note that both \( X \sqcup Y \) and \( X \bowtie Y \) also belong to \( \text{In}(\mathcal{K}) \) since a subset of an independent set is again independent.

A function \( f : \text{In}(\mathcal{K}) \to \mathbb{R} \) is **b-submodular** if \( f(\emptyset) = 0 \) and

\[
f(X) + f(Y) \geq f(X \lor Y) + f(X \bowtie Y)
\]

for \( X, Y \in \text{In}(\mathcal{K}) \).

**Theorem 15.1** ([115]) In case of a poset convex geometry, Faigle–Kern’s dual greedy algorithm (G) provides an optimal solution of (D) if and only if \( f \) is b-submodular.

\( \chi^A \in \{0, 1\}^E \) denotes a characteristic vector of \( A \subseteq E \), i.e. \( \chi^A(e) = 1 \) if \( e \in A \) and otherwise \( \chi^A(e) = 0 \).

For a convex geometry, a function \( f : \text{In}(\mathcal{K}) \to \mathbb{R} \) is **c-submodular** if \( f(\emptyset) = 0 \) and

\[
f(X) + f(Y) \geq f(X \lor Y) + f(X \sqcup Y) + f(X \bowtie Y)
\]

for \( X, Y \in \text{In}(\mathcal{K}) \) such that

\[
\chi^X + \chi^Y = \chi^{X \lor Y} + \chi^{X \sqcup Y} + \chi^{X \bowtie Y}.
\]

**Theorem 15.2** ([103]) For a convex geometry, Faigle–Kern’s dual greedy algorithm (G) provides an optimal solution of (D) if and only if \( f \) is c-submodular.
15.2 Max-Flow Min-Cut Property of 2-dimensional affine convex geometry

$(E, \mathbb{R}^2)$ を 2 次元平面上の点配置、$z \in E$ をひとつの選び、$E' = E \setminus z$ とする。この点配置の定める凸何を $(K, F, E)$ として、$z$ を根としたこの stem clutter を $C(z)$、bond clutter を $D(z)$ とする。

$z$ を中心とした単位円を $S_z^1$ とする。各 $e \in E'$ に対して $z$ を端点として $e$ を通る無限半直線が $S_z^1$ と交わる点を $\varphi_z(e)$ とし、$\varphi_z(z) = z$ として像写像 $\varphi_z : E \to S_z^1$ を定める。$T = \varphi_z(E')$ とおくと、$\varphi_z$ は $T \cup z$ 上で全射で、かつ $|\varphi_z^{-1}(z)| = 1$ である。

各 $S_z^1$ 上の点 $t$ に対して

$$\deg_z(t) = |\varphi_z^{-1}(t)|$$

を $t$ の $\varphi_z$ に関する重複度と呼ぶ。

$(T \cup z, \mathbb{R}^2)$ が 2 次元平面上の点配置として定める凸何・アンチトロイドを $(K_T, F_T, T \cup z)$ として、根を $z$ としたときの stem clutter, bond clutter をそれぞれ $C_T$, $D_T$ とする。$S_z^1$ を原点を中心とした単位円 $S^1$ に同一視すれば、$T$ は円周上の点配置であり、これの定める $T$ 上の有向トロイドを考えると、この有向トロイドの positive circuits の全体と $C_T$ は同じになる。さらに詳しく言えば、全射 $\varphi_z : E \to S_z^1$ に伴う $(K_T, F_T, T \cup z)$ の増幅 $(\varphi_z(K_T), \varphi_z(F_T), E)$ を考えると、凸何・アンチトロイドとしては $(K, F, E)$ とは一般に異なるが、$C(z)$, $D(z)$ は $C_T$, $D_T$ の複製と二重化になっている。つまり、$C(z)$ は $C_T$ の $\varphi_z$ による複製であり、$D(z)$ は $D_T$ の $\varphi_z$ による二重化になっている。

Def. 15.3 単位円周 $S_z^1$ 上の点 $t$ の対面点 (the antipodes of t) を $\tilde{t}$ で表す。つまり $\tilde{t}$ は、$\frac{1}{2}t + \frac{1}{2} \tilde{t} = z$ をみたす点である。

Def. 15.4 $S_z^1$ 上の 2 点 $t_1, t_2$ に対して $[t_1, t_2]$ を $t_1$ を始点として $t_2$ を終点として時計回りに円周 $S_z^1$ をたどった弧で、端点 $t_1$ を含み、$t_2$ を含まないものとする。$(t_1, t_2)$ を端点 $t_1$ を含み $t_2$ を含まない弧とする。同様に $[t_1, t_2]$, $(t_1, t_2)$ を定める。

Def. 15.5 $S_z^1$ の部分集合 $A$ に対して、その重複度総数を

$$\text{Deg}(A) = \sum_{\alpha \in A} \deg_z(\alpha)$$

とおく。

Def. 15.6 ある点 $t \in S_z^1$ に対して $[t, -t]$ または $(t, -t)$ と書ける弧を半円弧と呼ぶこととする。

半円弧 $C \subseteq S_z^1$ に対して

$$H = \{ x \in \mathbb{R}^2 \setminus \{z\} : \varphi_z(x) \in C \}$$

で定まる集合を (z を根とした) 半空間と呼ぶ。$\tilde{C} = \{ \tilde{t} : t \in C \}$, $\tilde{H} = \{ \varphi_z(x) \in \tilde{C} \}$ すれば、$H$ と $\tilde{H}$ は $\mathbb{R} \setminus z$ を分割している。

Def. 15.7 ある半空間 $H$ に対して $H \cap E'$ と書ける集合を residual set と呼び、res($H$) = $|H \cap E'|$ をその residual number と呼ぶことにする。$H$ が半円弧 $C$ の拡張の半空間であれば、自明に res($H$) = Deg($C$) = $\sum \deg_z(\alpha) : \alpha \in C$ である。

Lemma 15.8 residual sets の中の極小元の全体は、$D(z) = b(C(z))$ に一致する。特に、residual number の最小値と $C(z)$ の blocking number は一致する。
定理 15.9 $\mathbb{R}^2 \setminus z$ 中の 5 点 $u_1, \ldots, u_5$ が Pentagon 配置をなすとは以下を満たすこととする。

1. $\varphi_z(u_1), \ldots, \varphi_z(u_5)$ がこの順に (clockwise に) $S_2^1$ 上に並んでいる。
2. 各 $i = 1, \ldots, 5$ に対して

$$\varphi_z(u_i) \in (\varphi_z(u_{i+2}), \varphi_z(u_{i+3}))$$

である。（ただし、添え字は mod 5 で数える。）

$E$ が $\mathbb{R}^2$ 中の点配置であることからの、$C(z)$ の元のサイズは 2 または 3 であることは明らか。ここで、パッキングを考えるときには、サイズ 2 の元は無視してもよいことをまず示そう。

レモニオン 15.10 $\mathcal{L} \subseteq C(z), X \in \mathcal{L}, |X| = 2$ とする。このとき、$\mathcal{L}$ がパックするための必要十分条件は、$\mathcal{L}' = \mathcal{L} \setminus X$ がパックすることである。
（ここで \setminus は clutter の deletion で、\{X\} という意味ではないことに注意。）

コロリー 15.11 $C(z)$ からサイズ 2 の元をすべて除いた clutter を $C'$ とすると、$C(z)$ がパックすることと、$C'$ がパックすることは必要十分になる。

メインの補題を述べる。

レモニオン 15.12 $C(z)$ がパックせずにかつ任意の proper subset $X \subseteq E'$ に対して $C(z)|X$ がパックするとき、$E'$ は $z$ を中心とした Pentagon 配置を含む。

$C(z)$ の任意の proper な contraction–minor は、affine kernel–shelling の凸幾何学の stem clutter に、定理（八森）から常にパックする。ゆえに、

レモニオン 15.13 以下は同値になる。

1. $C(z)$ とそのすべての deletion–minor はパックする。

2. $C(z)$ とそのすべてのマイナーはパックする。

このと補題 15.12 と補題 15.13 から、

テオレミ 15.14 以下は同値になる。

1. $C(z)$ とそのすべてのマイナーはパックする。

2. $C(z)$ は $T_3^5$ をマイナーとして含まない。

3. $C(z)$ は $T_3^5$ を deletion–minor として含まない。

4. $E'$ の部分集合で $z$ を中心とした Pentagon 配置になるものは存在しない。

5. $C(z)$ とそのすべての deletion–minor はパックする。

（Proof）(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) は、自明。
(4) $\Rightarrow$ (5) は補題 15.12 よりわかる。

(5) $\Rightarrow$ (1) は補題 15.13 より。

$C_T$ の任意の複製はある $C(z)$ として実現できるから、「$C_T$ が packing property を持てば、その任意の複製もパックする」という replication lemma が $C_T$ に対して成り立つことが、上の定理で示されたことによる。
Def. 15.15 有向マトロイド $\mathcal{M} = (\mathcal{C}, \mathcal{D}; E)$ で、$\{X : (X, \emptyset) \in \mathcal{C}\}$ をその positive circuits 全体のなす clutter と呼ぶ。

有向マトロイドが有向グラフから定義されるとき、positive circuits の全体は、有向閉路の全体になり、その blocker は feedback arc set と呼ばれる。最小 feedback arc set 問題は NP-hard だが、グラフが平面グラフであれば、多項式で解けると知られている。グラフが無向グラフのときは、feedback arc set は補木にほかならないので、自明な問題になる。

$S_1^1$ を $S^1$ に同一視して考えたときの点配置 $T$ の定める有向マトロイドの positive circuits のなす clutter は $C_T$ と等しい。

ゆえに、点の複製を考えなければ、これまでの問題は、有向マトロイドの positive circuit のなす clutter のパッキング問題であったともみなせる。このとき、予想として

Conjecture 15.16 (八森) ランク $d$ の有向マトロイドで positive circuit のサイズがすべて $d + 1$ であるとき、positive circuit clutter が packing property を持つための必要十分条件は、台集合が $d + 1$ 個の集合 $E_1, E_2, \ldots, E_{d+1}$ に分割できて各 $(E_i, E - E_i)$ ($i = 1, \ldots, d + 1$) がコベクトルになることである。
Chapter 16

Greedy algorithms

16.1 Greedoids

Greedoids という概念は Korte and Lovász によって 1980 年前後に、交換律、半交換律、greedy algorithm などの諸概念をひろく包括する framework として創案されたものである。It includes as a special case both matroids and antimatroids as well as various decomposition schemes such as Gaussian elimination of matrices, ear decomposition of 2-connected graphs, bisimplicial elimination in bipartite graphs, series-parallel reduction of graphs, retract elimination in directed graphs, etc.

The systematic studies of antimatroids seem to be started by Edelman [59] and Jamison [96]. (See also Edelman and Jamison [63].) They studied a convex geometry, which is the complement of an antimatroid, as an abstraction of convexity. Matroids and antimatroids are 'dual' or 'antipodal' to each other in the sense that the closure of a matroid has the Steinitz-MacLane exchange property, while that of an antimatroid satisfies the anti-exchange law. Antimatroids typically arise as a collection of allowed sequences of deletion or scanning on a combinatorial object such as poset shelling, node or edge shelling of trees, simplicial elimination sequences in triangulated graphs, vertex shelling of polytopes, node-search or line-search on graphs, and so on. For more details of greedoids and antimatroids, we refer to [112].

In the algorithmic aspect, a linear objective function can be optimized by a greedy algorithm on matroids, and conversely this property characterizes matroids. In contrast with this, optimizing a linear function on greedoids is an intractable problem.

Goecke, Korte and Lovász [79] は、の最大化問題が greedy algorithm で解けることが、greedoid を特徴づけることを示した。

一方、Boyd and Faigle [35] は、'nested bottleneck function' の最適化問題がアンチマトロイド上では greedy algorithm で解け、かつそれがアンチマトロイドの特徴付がになることを示した。Boyd and Faigle's result is a mere extension of Lawler's one [116] that a single-machine scheduling problem obeying a precedence constraint of a partial order among jobs can be solved by a greedy procedure when the objective is a certain bottleneck function.

greedoids と antimatroids は simple languages として、つまり同じ要素を 2 度以上含まない言語として定式化されている。それに対して、Björner and Ziegler [22] は three greedoids を non-simple languages に拡張する可能性について 3 つのタイプを挙げて考察している。

一方、Björner, Lovász and Shor [26] は、chip-firing games of graphs の研究から non-simple な言語としてのアンチマトロイドを導入して考察した。この non-simple antimatroid を以下本稿では poly-antimatroid
16.2 Greedoids and antimatroids

Let $E$ denote a finite non-empty set throughout this note. A word on $E$ is a finite sequence of elements of $E$, and $\epsilon$ denotes a word of null length. For a word $\alpha = a_1 \cdots a_k$, $\tilde{\alpha}$ denotes the set of elements of $\alpha$, and $|\alpha|$ is its length. A set of words is called a language. For the sake of simplicity, we assume in this paper that languages are all finite. For a language $L$, $\rho(L)$ is the maximal length of words of $L$. If every maximal word has the same finite length, $L$ is called pure. A word is simple if it has no repeated elements, and a language is called simple if its words are all simple. A language is non-simple if it is not restricted to be simple words. That is, 'non-simple' implies 'not necessarily simple' throughout this paper. A language $L$ is called left-hereditary if $\epsilon \in L$ and a left-prefix of its word belongs to $L$ again, i.e. $\alpha \beta \in L$ implies $\alpha \in L$.

For a fixed number $r$, $\{\alpha \in L : |\alpha| \leq r\}$ is called a truncation of $L$.

Let $L$ be a left-hereditary simple language on $E$. $L$ is called a greedoid if it satisfies

$$\text{(Gr)} \quad \alpha, \beta \in L, |\alpha| > |\beta| \implies \exists x : x \in \tilde{\alpha} \setminus \tilde{\beta}, \beta x \in L.$$ 

And $L$ is called an antimatroid if it satisfies

$$\text{(An)} \quad \alpha, \beta \in L, \tilde{\alpha} \not\subseteq \tilde{\beta} \implies \exists x : x \in \tilde{\alpha} \setminus \tilde{\beta}, \beta x \in L.$$ 

As is easily seen, a greedoid and an antimatroid are simple pure languages. Although a greedoid as well as an antimatroid can be formulated both as a language (ordered version) and as a family of sets (unordered version), we treat in this paper only with languages, i.e. the ordered version.

Let $L$ be a simple or non-simple language on $E$, and suppose further $L$ is left-hereditary and pure. Let $F$ be a real-valued function on $L$. Then associated with $F$, we can define a function

$$w(\alpha) = \min \{ F(a_1 a_2 \cdots a_i) : i = 1, \ldots, k \} \quad \text{for } \alpha = a_1 \cdots a_k \in L. \quad (16.1)$$

And we shall consider the optimization problem below:

$$\max_{\alpha \in L, |\alpha| = \rho(L)} w(\alpha) \quad \text{subject to } \alpha \in L, \rho(L) \quad (16.2)$$

The following procedure gives a candidate of solution of (16.2).

< GREEDY >

$\alpha \leftarrow \epsilon$ ;

while $\Gamma(\alpha) \neq \emptyset$ and $|\alpha| < \rho(L)$ do begin

choose $x \in \Gamma(\alpha)$ such that $w(\alpha x)$ is maximum ;

$\alpha \leftarrow \alpha x$ ;

end

where $\Gamma(\alpha)$ is the set of elements which succeed $\alpha$, i.e. $\Gamma(\alpha) = \{ x \in E : \alpha x \in L \}$. 
We shall first describe the algorithmic characterization of greedoids. Let \( Z_+ \) be the set of nonnegative integers, and \( f \) a real-valued function on \( E \times Z_+ \) such that it is monotone with respect to the second variable, i.e.

\[
i \leq j \implies f(e, i) \leq f(e, j)
\]

(16.3)

Setting \( F(a_1 \cdots a_i) = f(a_i, i) \) in (16.1) gives rise to a bottleneck function such that

\[
w(\alpha) = \min \{ f(a_i, i) : i = 1, \ldots, k \} \quad (\alpha = a_1 \cdots a_k \in L)
\]

(16.4)

which is called a generalized bottleneck function. Then we have

**Theorem 16.1 (Goecke, Korte and Lovász [79])** If \( L \) is a greedoid, then GREEDY gives an optimal solution for a generalized bottleneck function. Conversely, for a left-hereditary simple pure language \( L \), if the solution given by GREEDY is always optimal for any generalized bottleneck function, then \( L \) is a greedoid.

We have a similar algorithmic characterization for antimatroids as well, the result of which has the origin in the work of Lawler [116] on a single-machine scheduling problem. He showed that when a precedence constraint is given as a partial order among jobs and the objective function to be minimized is a certain bottleneck function, a single-machine scheduling problem can be solved by a greedy procedure. Extending this, Boyd and Faigle [35] have shown that an analogous result holds for antimatroids and in this case the converse is also true. Let \( f \) be a real-valued function on \( E \times 2^E \). And suppose \( f \) to be monotone with respect to the second variable, i.e.

\[
A \subseteq B \implies f(e, A) \leq f(e, B).
\]

(16.5)

Setting \( F(a_1 \cdots a_i) = f(a_i, \{a_1, \ldots, a_i\}) \) in (16.1), we have a function on \( L \) such that

\[
w(\alpha) = \min_{i=1, \ldots, k} \{ f(a_i, \{a_1, a_2, \ldots, a_i\}) \} \quad (\alpha = a_1 \cdots a_k \in L),
\]

(16.6)

which is called a nested bottleneck function. Then,

**Theorem 16.2 (Boyd and Faigle [35])** If \( L \) is a truncation of an antimatroid, the algorithm GREEDY gives an optimal solution for a nested bottleneck function. Conversely, for a left-hereditary simple pure language \( L \), if the solution given by GREEDY is always optimal for any nested bottleneck function, then \( L \) is a truncation of an antimatroid.

### 16.3 Polygreedoids

To describe the definitions of polygreedoids and ply-antimatroids, we prepare some terminology. For a vector \( v \in \mathbb{R}^E \), \( |v| \) denotes its 1-norm \( \sum(|v_i| : i \in E) \). \( u \lor v \in \mathbb{R}^E \) denotes a join of vectors, i.e. \( (u \lor v)_e = \max\{u_e, v_e\} \). For a word \( \alpha = a_1a_2 \cdots a_k \), a subsequence \( \alpha' = a_{i_1}a_{i_2} \cdots a_{i_m} \) such that \( 1 \leq i_1 < i_2 < \ldots < i_m \leq k \) is a subword of \( \alpha \), and we write it as \( \alpha' \subseteq \alpha \). Let us denote by \( \alpha_x \) the number of repetitions of an element \( x \) in \( \alpha \), and \( [\alpha]_e = \alpha_e (e \in E) \), which is the score vector of \( \alpha \). An ascending chain of integral points \( u^{(0)} \leq u^{(1)} \leq \ldots \leq u^{(m)} (u^{(i)} \in \mathbb{Z}^E) \) is elementary if for \( i = 1, \ldots, m \), there exists \( j \in E \) such that \( u^{(i)}_j = u^{(i-1)}_j - 1 \) for \( j' \in E \) with \( j' \neq j \) and \( u^{(i)}_j = u^{(i-1)}_j + 1 \).
Björner and Ziegler [22] investigated three possible extensions of greedoids to non-simple languages such as

\[(NG) \quad \alpha, \beta \in L, \ |\alpha| > |\beta| \implies \exists x \in E : x \in \alpha \text{ and } \beta x \in L.\]

\[(SG) \quad \alpha, \beta \in L, \ |\alpha| > |\beta| \implies \exists \text{ subword } \alpha' \text{ of } \alpha : \beta \alpha' \in L \text{ and } |\alpha'| = |\alpha| - |\beta|.\]

\[(PG) \quad \alpha, \beta \in L, \ |\alpha| > |\beta| \implies \exists x \in E : |\alpha|_x > |\beta|_x \text{ and } \beta x \in L.\]

where \(L\) is a left-hereditary non-simple language. They called \(L\) a ’(non-simple) greedoid’ if it satisfies (NG), a ’strong greedoid’ if it satisfies (SG), and a ‘polygreedoid’ if it satisfies (PG), respectively. They have mentioned that the conditions (SG) and (PG) are independent and that the reduced expressions of a finite Coxeter group gives rise to a strong greedoid (SG).

Instead of strong greedoids, in this paper, we shall consider polygreedoids as the definition of non-simple versions of greedoids. That is, we shall call a left-hereditary non-simple language a polygreedoid if it satisfies (PG).

Let \(f\) be a real-valued function on \(E \times \mathbb{Z}_+ \times \mathbb{Z}_+\), and suppose it satisfies

\[j = j', \ i \leq i' \implies f(e, j, i) \leq f(e, j', i').\]  

(16.7)

And for \(\alpha = a_1 \cdots a_k\), we set \(F(\alpha) = f(a_k, [\alpha]_a_k, k)\). Then a generalized bottleneck function of (16.4) is generalized to

\[w(\alpha) = \min_{i=1, \ldots, k} \{F(a_1 \cdots a_i)\} = \min_{i=1, \ldots, k} \{f(a_i, [\alpha]_{a_i}, i)\},\]  

(16.8)

which we call a **doubly generalized bottleneck function**. Then Theorem 16.3 below holds, which seems to justify our choice of (PG) as the definition of polygreedoids.

**Theorem 16.3** Let \(L\) be a finite non-simple language. If \(L\) is a polygreedoid, then for a doubly generalized bottleneck function, the algorithm GREEDY gives an optimal solution. Conversely, suppose \(L\) is left-hereditary and pure, and if the solution given by GREEDY is necessarily optimal for any doubly generalized bottleneck function, then \(L\) is a polygreedoid.

(Proof) We shall prove the first half. Let \(\alpha = a_1 \cdots a_k\) be the \(k\)-th intermediate solution of GREEDY. We use induction on \(k\). Suppose \(\alpha\) attains the maximum of \(w\) among the words of length \(k\) in \(L\), and \(a_k a_{k+1}\) to be the \((k+1)\)-th solution of GREEDY. And suppose, contrarily, \(a_k a_{k+1}\) is not optimal, and \(\beta = b_1 \cdots b_k b_{k+1}\) be an optimal solution of length \(k + 1\). Then we have \(w(\beta) > w(a_k a_{k+1})\). Hence,

\[\min_{i=1, \ldots, k+1} \{f(b_i, [\beta]_{b_i}, i)\} > \min_{i=1, \ldots, k+1} \{f(a_i, [\alpha]_{a_i}, i)\}.\]

If the minimum of the right-hand side is attained for some \(i\) with \(1 \leq i \leq k\), then we have

\[\min_{i=1, \ldots, k} \{f(b_i, [\beta]_{b_i}, i)\} \geq \min_{i=1, \ldots, k+1} \{f(b_i, [\beta]_{b_i}, i)\} > \min_{i=1, \ldots, k+1} \{f(a_i, [\alpha]_{a_i}, i)\} = \min_{i=1, \ldots, k} \{f(a_i, [\alpha]_{a_i}, i)\},\]

which contradicts the optimality of \(\alpha = a_1 \cdots a_k\). Hence the minimum is attained at \(i = k + 1\) and we have

\[\min_{i=1, \ldots, k+1} \{f(b_i, [\beta]_{b_i}, i)\} > f(a_{k+1}, [\alpha]_{a_{k+1}}, k + 1).\]
Since $|\beta| > |\alpha|$, (PG) implies that there exists $x \in \tilde{\beta}$ such that $\beta_x > \alpha_x$ and $\alpha \in L$. Hence $x$ appears in $\beta$ more than $\alpha_x$ times. So let $b_j$ be the $(\alpha_x + 1)$-th $x$ in the sequence $\beta$. Then, from the monotonicity of $f$, we have
\[
F(ab_j) = F(\alpha x) = f(x, \alpha x + 1, k + 1) \\
\geq f(x, \alpha x + 1, j) = F(b_1 \cdots b_j) \\
\geq \min_{i=1, \ldots, k+1} \{F(b_1 \cdots b_i)\} > F(\alpha a_{k+1})
\]
which contradicts the choice of $a_{k+1}$. Hence the proof of the first half is completed.

We shall describe the proof of the second half. Suppose, contrarily, that (PG) does not hold for $\alpha = a_1 \cdots a_n, \beta = b_1 \cdots b_m \in L$ with $|\alpha| > |\beta|$. Take $k$ to be the minimal index such that $[a_1 \cdots a_k] > \beta_a$, and let $x = a_k$. By assumption such $k$ necessarily exists. Let us define a function on $E \times \mathbb{Z}_+ \times \mathbb{Z}_+$ such that
\[
f(e, j, i) = \begin{cases} 1 & \text{if either } j \leq \beta_e \text{ or } i > \beta_x \\ 0 & \text{otherwise} \end{cases}
\]
It is easy to see that $f$ satisfies (16.7). Using this $f$, let us define
\[
F(y_1 \cdots y_p) = f(y_p, [y_1 \cdots y_p]_{y_p}, [y_1 \cdots y_p]_x) \quad \text{for } y_1 \cdots y_p \in L,
\]
and $w$ be the associated bottleneck function.

Then $w(\alpha) = 1$ holds. Actually, for $i = 1, \ldots, k - 1$, we have $[a_1 \cdots a_i] \leq \beta_a$, from the choice of $k$, and hence $F(a_1 \cdots a_i) = 1$. For $i = k, \ldots, n$, $[a_1 \cdots a_i] \geq [a_1 \cdots a_k]_x > \beta_x$ and so $F(a_1 \cdots a_i) = 1$. Hence $w(\alpha) = \min\{F(a_1 \cdots a_i) : i = 1, \ldots, n\} = 1$. In particular, this implies that $\alpha$ is an intermediate solution and will be extended to a final solution $\alpha^1$ of GREEDY. A similar argument shows $w(\alpha^1) = 1$.

For $j = 1, \ldots, m$, we have $f(b_j, [b_1 \cdots b_j]_{b_j}, [b_1 \cdots b_j]_x) = 1$ since $[b_1 \cdots b_j]_{b_j} \leq \beta_{b_j}$ trivially holds. Hence $w(\beta) = 1$, and $\beta$ is also an intermediate solution of GREEDY, which will be extended to a final solution $\beta^1$. Suppose $z \in E$, $\beta_z \in L$. Then by assumption, $z \neq x$. Since $[\beta z]_z = \beta z + 1 > \beta_z$ and $[\beta z]_x = \beta_x$, $f(z, [\beta z]_z, [\beta z]_x) = 0$. From the definition of $w$, $w(\beta^1) = 0$ follows. Hence $w(\alpha^1) \neq w(\beta^1)$, which contradicts the optimality of the solutions by GREEDY. And the proof is completed. \[\square\]

### 16.4 Poly-antimatroids

Björner, Lovász and Shor [26] proposed that a poly-antimatroid is defined as a language such that it is left-hereditary, locally free and permutable. That is, let $L$ be a left-hereditary non-simple language on $E$.

Then $L$ is said to be a **poly-antimatroid** if it holds that

- (LF) $\alpha x, \alpha y \in L$, $x \neq y$ $(x, y \in E) \implies \alpha y x \in L$. \hspace{1cm} [locally free]
- (PM) $\alpha, \beta \in L$, $[\alpha] = [\beta]$, $\alpha x \in L$ $(x \in E) \implies \beta x \in L$. \hspace{1cm} [permutable]

The pair of conditions (LF) and (PM) is easily seen to be equivalent to the exchange property (EX) below.

- (EX) $\alpha x, \beta \in L$, $[\alpha] \leq [\beta]$, $[\alpha x] \not\in [\beta]$, $(x \in E)$, $\implies \beta x \in L$

Furthermore, (EX) is equivalent to the stronger exchange property (StEX). (See [26].)
Let us denote by \( \mathcal{N} = \{ [\alpha] : \alpha \in L \} \) the set of score vectors of \( L \). It follows immediately from the above properties of \( L \) that \( \mathcal{N} \) satisfies the following:

1. \( 0 \in \mathcal{N} \),
2. For any \( v \in \mathcal{N} \), there exists an elementary chain in \( \mathcal{N} \) from the origin \( 0 \) to \( v \),
3. if \( u, v \in \mathcal{N} \), then \( u \lor v \in \mathcal{N} \).

In particular, with respect to the ordinary partial order among vectors, \( \mathcal{N} \) constitutes a locally free lattice. Hence, it is isomorphic to a lattice of feasible sets of a certain simple antimatroid.

Conversely, we shall call a finite collection of non-negative integral points a score space if it satisfies the above three conditions. Every score space is derived from a poly-antimatroid. In fact,

\[
L_{\mathcal{N}} = \{ a_1 \cdots a_k : a_i \in E, \ 0 \leq [a_1] \leq [a_1 a_2] \leq \cdots \leq [a_1 \cdots a_k] \text{ is a maximal chain of } \mathcal{N} \} \quad (16.10)
\]

is a poly-antimatroid whose set of score vectors equals to \( \mathcal{N} \). Hence the notions of poly-antimatroid and score space are seen to be equivalent.

A score space has the strong accessibility mentioned below.

**Proposition 16.4** Let \( u, v \in \mathcal{N} \) and \( u \leq v \). Then there exists an elementary chain in \( \mathcal{N} \) connecting \( u \) and \( v \).

(Proof) From \( v \in \mathcal{N} \), there exists an elementary chain \( \{ v_i \} \) from the origin to \( v \). Then \( \{ u \lor v_i \} \) is a chain in \( \mathcal{N} \) possibly including repeated elements, which contains obviously an elementary chain between \( u \) and \( v \) as a subsequence. \( \square \)

This can be restated in terms of language as

**Corollary 16.5** Let \( L \) be a poly-antimatroid, \( \alpha, \beta \in L \), and suppose \([\alpha] \leq [\beta]\). Then there exists a subword \( \gamma \) of \( \beta \) such that \( \alpha \gamma \in L \) and \([\alpha \gamma] = [\beta]\).

Let \( L \) be a poly-antimatroid, and \( \mathcal{N} \) be the score space of \( L \). Suppose \( f \) to be a real-valued function on \( E \times \mathcal{N} \) such that

\[
u_e = v_e, \ u \leq v \implies f(e, u) \leq f(e, v), \quad (16.11)
\]

Then a nested bottleneck function of (16.6) is generalized to

\[
w(\alpha) = \min \{ f(a_i, [a_1 \cdots a_i]) : i = 1, \ldots, r \} \quad (\alpha = a_1 a_2 \cdots a_r \in L), \quad (16.12)
\]

which we shall call a score-nested bottleneck function. Then Theorem 16.2 is extended to poly-antimatroids as follows.

**Theorem 16.6** For a poly-antimatroid and a score-nested bottleneck function, the solution by GREEDY is necessarily optimal. And for a left-hereditary non-simple pure language \( L \), if the greedy solution is optimal for any score-nested bottleneck function, then \( L \) is a truncation of a poly-antimatroid.
(Proof) Let \( \alpha = x_1 \cdots x_k \) be the greedy solution after the \( k \)-th iteration of GREEDY. We use induction on \( k \). Suppose that the assertion holds until \( k \), and let \( \alpha x_{k+1} \) be the greedy solution of length \( k + 1 \). Suppose there exists an optimal solution \( \beta y_{k+1} = y_1 \cdots y_k y_{k+1} \in L \) of length \( k + 1 \) and \( w(\beta y_{k+1}) > w(\alpha x_{k+1}) \). By the assumption and the induction hypothesis, we have

\[
\min_{i=1, \ldots, k+1} \{ f(y_i, [y_1 \cdots y_i]) > f(x_{k+1}, [x_1 \cdots x_{k+1}]) \} \tag{16.13}
\]

If \( \alpha = \beta \) holds, then from the definition of GREEDY, we have

\[
w(\beta y_{k+1}) > w(\alpha x_{k+1}) \geq w(\alpha y_{k+1}) = w(\beta y_{k+1}),
\]

which is a contradiction.

Hence we have \( \alpha \neq \beta \). Then there exists \( p \) with \( 1 \leq p \leq k \) such that

\[
[y_1 \cdots y_{p-1}] \leq [\alpha], \quad [y_1 \cdots y_p] \not\leq [\alpha].
\]

By (EX), we have \( \alpha y_p \in L \).

Clearly, \( [\alpha y_p] \geq [y_1 \cdots y_p] \) and \([\alpha y_p]_{y_p} = [y_1 \cdots y_p]_{y_p} \). Hence from the 2-monotonicity of \( f \), we have

\[
f(y_p, [\alpha y_p]) \geq f(y_p, [y_1 \cdots y_p]) \geq \min_{i=1, \ldots, k+1} \{ f(y_i, [y_1 \cdots y_i]) \} > f(x_{k+1}, [\alpha x_{k+1}]),
\]

which contradicts the choice of \( x_{k+1} \). The proof of the first half is completed.

Conversely, let \( L \) be a left-hereditary non-simple language and \( \rho(L) < +\infty \). Suppose GREEDY always gives an optimal solution.

We shall first prove that \( L \) is locally free. Let \( \alpha \in L \) and \( [\alpha] < \rho(L) - 1 \). Suppose \( \alpha x, \alpha y \in L, x \neq y \) and \( \alpha x y \notin L \). Let us define

\[
f(e, v) = \begin{cases} 1 & \text{if } v_e \leq [\alpha y]_e \text{ or } v_x > \alpha_x \\ 0 & \text{otherwise} \end{cases} \quad (e \in E, v \in \mathbb{Z}^E)
\]

We shall show that \( f \) satisfies (16.11). Suppose, contrarily, \( v_e = v'_e, v \leq v' \), and \( 1 = f(e, v) > f(e, v') = 0 \). Then either \( v_e \leq [\alpha y]_e \) or \( v_x > \alpha_x + 1 \) holds, and both \( v'_e > [\alpha y]_e \) and \( v'_x < \alpha_x + 1 \) hold. These give a contradiction. Hence \( f \) satisfies (16.11). For this \( f \), we have \( w(\alpha x) = w(\alpha y) = 1 \). So \( \alpha x, \alpha y \) arise as an intermediate solution of GREEDY, and they are extended to the final solutions denoted \( \alpha^1 \) and \( \alpha^2 \), respectively.

Let \( \alpha^1 = \alpha x u_1 \cdots u_q \). For \( i = 1, \ldots, q \), \([\alpha x u_1 \cdots u_i]_x > \alpha_x \) and \( f(u_i, \alpha x u_1 \cdots u_i) = 1 \) hold. Hence \( w(\alpha^1) = 1 \). Next we shall show \( w(\alpha^2) = 0 \). Suppose \( \alpha y z \in L \) for \( z \in E \). By assumption, \( z \neq x \). Trivially, \([\alpha y z]_z \geq [\alpha y]_z + 1 > [\alpha y]_z \) and since \( y \neq x \) and \( z \neq x \), we have \([\alpha y z]_x = \alpha_x \). Hence \( f(z, \alpha y z) = 0 \), and \( w(\alpha^2) = 0 \) follows. Since \( w(\alpha^1) \neq w(\alpha^2) \), this contradicts the optimality of the algorithm GREEDY.

Next we shall show the permutability of \( L \). Let \( \alpha, \beta \in L \) and \( [\alpha] = [\beta] < \rho(L) \). Suppose \( [\alpha] = [\beta], \alpha x \in L \) and \( \beta x \notin L \). Then

\[
f'(e, v) = \begin{cases} 1 & \text{if } v_e \leq \alpha_e \text{ or } v_x > \alpha_x \\ 0 & \text{otherwise.} \end{cases}
\]

is a function satisfying (16.11). Let us first show \( w(\alpha x) = 1 \) and \( w(\beta) = 1 \). Suppose \( \alpha = a_1 \cdots a_n \). For \( i = 1, \ldots, n \), since \([a_1 \cdots a_i]_a_i \leq [\alpha]_a_i \) is trivially satisfied, we have \( f'(a_i, a_1 \cdots a_i) = 1 \). Since \([\alpha]_x > \alpha_x \),
we have \( f(x, ax) = 1 \). Hence \( w(ax) = 1 \) follows. Suppose \( \beta = b_1 \cdots b_n \). Since \([b_1 \cdots b_i]_{b_i} \geq \beta]_{b_i} = [\alpha]_{b_i}
\) is obvious, \( f'(b_i, b_1 \cdots b_i) = 1 \) holds for \( i = 1, \ldots, n \). Hence \( w(\beta) = w(b_1 \cdots b_n) = 1 \). Both \( ax \) and \( \beta \) are intermediate solutions of GREEDY, and they are extended to the final solutions denoted \( \alpha^1 \) and \( \beta^1 \), respectively.

Suppose \( \alpha^1 = axd_1 \cdots d_r \). Then for any \( j = 1, \ldots, r \), we have \( f'(d_j, axd_1 \cdots d_j) = 1 \) since \([axd_1 \cdots d_j]_x \geq [\alpha]_x \) obviously holds. Hence we have \( w(\alpha^1) = 1 \). Next we shall show \( w(\beta^1) = 0 \). Suppose \( \beta z \in L \) for \( z \in E \). From assumption, \( z \neq x \). Then \([\beta z]_z = \beta z + 1 = \alpha z + 1 \notin \alpha x \) and \([\beta z]_x = \beta x = \alpha x \neq \alpha z \), which gives \( f(z, \beta z) = 0 \). Hence by definition, \( w(\beta^1) = 0 \). This contradicts the optimality of GREEDY, and the proof is completed. □

As a concluding remark, we shall present a variant of Theorem 16.6. Replacing the partial order among score vectors in the exchange property (EX) with that of subword-inclusion provides a condition

\[
\alpha x, \beta \in L, \; \alpha \subseteq \beta, \; \alpha x \nsubseteq \beta \implies \beta x \in L.
\]

We can consider it as an upper interval property over non-simple words. If a non-simple language \( L \) is left-hereditary and satisfies (UI), let us call it an upper interval language. As is easy to observe, an upper interval language is locally free, but not permutable in general.

A complete analogue of Theorem 16.6 holds for upper interval languages. Let \( f \) be a real-valued function on \( E \times L \), and suppose \( f \) satisfies

\[
\alpha_e = \alpha'_e, \; \alpha \subseteq \alpha' \implies f(e, \alpha) \leq f(e, \alpha').
\]

This gives rise to a bottleneck objective function

\[
w(\alpha) = \min_{i=1, \ldots, r} \{ f(a_i, a_1 \cdots a_i) : i = 1, \ldots, r \},
\]

and we have

**Theorem 16.7** Let \( L \) be a non-simple language which is left-hereditary and pure. If \( L \) is a truncation of an upper interval language, then GREEDY gives an optimal solution for any function of (16.17). Conversely, if the solution given by GREEDY is always optimal, then \( L \) is a truncation of an upper interval language.

(Proof) The proof is completely analogous to that of Theorem 16.6. □
Chapter 17

Implicational Systems

17.1 Congruence and anti-exchange property

An operator \( f \) satisfies the congruence relation if \( f(A) = f(B) \) implies \( f(A \cup C) = f(B \cup C) \) for any \( C \).

Proposition 17.1 Every path-independent operator \( f \) satisfies the congruence relation.

(Proof) \( f(A) = f(B) \) implies that \( f(A \cup C) = f(f(A) \cup C) = f(f(B) \cup C) = f(B \cup C) \).

An arbitrary operator \( f \) gives a relation \( A \vdash_f B \) by the definition: \( A \vdash_f B \) if \( f(A) = f(A \cup B) \). Recall the definition of an inference relation in Section 17.2.

Proposition 17.2 ([48]) If an operator satisfies the congruence condition, the relation \( \vdash_f \) is an inference relation.

(Proof) (a) is obvious. (b) is to be proved. Suppose \( A \vdash_f B \), which is equivalent to \( f(A) = f(A \cup B) \). By the assumption of congruence, \( f(A \cup C) = f(f(A) \cup C) = f(f(B) \cup C) = f(B \cup C) \). Hence we have \( A \cup C \vdash_f B \cup C \). Lastly, Suppose \( f(A) = f(A \cup B) \) and \( f(B) = f(B \cup C) \). The congruence relation implies \( f(A \cup C) = f(A \cup B \cup C) = f(A \cup B) = f(A) \). Hence \( A \vdash_f C \). This completes the proof.

Proposition 17.3 (Soltan 1984, [125]) An expanding operator \( f : 2^E \to 2^E \) is path independent if and only if it is a closure function.

EXERCISE 17.4 Prove Proposition 17.3.

Suppose \( (\mathcal{X}, f) \) to be a closure space on \( E \). Then

\[
\sigma_f(A) = \{ a \in E : A \vdash_f a \} = \{ a \in E : f(A \cup a) = f(A) \}
\]

(17.1)

is a closure function. Furthermore \( \sigma_f \) is equal to \( f \).

Suppose conversely that an inference relation \( \vdash \) is given on \( E \). Then

Proposition 17.5 \( \sigma_\vdash(A) = \{ a \in A : A \vdash a \} \) is a closure function.
The extensively and the monotonicity are obvious. We shall show the idempotency. \( \sigma_r(A) \subseteq \sigma_r(\sigma_r(A)) \) is obvious. By definition, \( A \vdash \sigma_r(A) \) holds, and if \( \sigma_r(A) \vdash a \) then \( A \vdash a \) from the inference axiom. Hence \( \sigma_r(A) \vdash a \) implies \( A \vdash a \), from which it follows that 

\[ \sigma_r(\sigma_r(A)) = \{ a \in E : \sigma_r(A) \vdash a \} \subseteq \{ a' \in E : A \vdash a' \} = \sigma_r(A). \]

Now we have \( \sigma_r(\sigma_r(A)) = \sigma_r(A) \). This completes the proof. \( \square \)

**Theorem 17.6 (Danilov and Koshevoy [48])** Let \( f : 2^E \to 2^E \) be a contracting operator (or a choice function). Then the following are equivalent.

1. \( f \) is path-independent,
2. \( f \) meets the congruence condition,
3. \( \vdash f \) is an inference relation.

(Proof) By Proposition 17.1, \( (1) \Rightarrow (2) \) is already shown. Proposition 17.2 shows \( (2) \Rightarrow (3) \). \( (3) \Rightarrow (1) \) is left to be shown. For any \( A \), \( f(A) \vdash f(A) \) is trivial by definition of \( \vdash f \). By the property (b) of the axiom, \( f(A) \cup B \vdash f(A \cup B) \). Hence \( f(f(A) \cup B) = f(A \cup B) \), which implies the path-independence of \( f \). \( \square \)

An inference relation is *framed* if, for any \( A \subseteq E \), there exists the minimum subset of \( A \) such that \( fr(A) \vdash A \), which we denote by \( fr(A) \). \( fr(A) \) is a frame of \( A \).

**Proposition 17.7 ([48])** Let \( \vdash \) be an inference relation and \( \sigma \) be the corresponding closure operator. Then the following are equivalent.

1. \( \sigma \) satisfies the anti-exchange condition,
2. \( \vdash \) is framed.

(Proof) 証明をここへ。 \( \square \)

### 17.2 Inference structures and Armstrong's laws

We shall define an *inference structure* as a binary relation \( R \) on the power set of a finite set \( E \), i.e. \( R \subseteq 2^E \times 2^E \), which satisfies the following conditions. An inference structure is also called a *full implicational system*. We write \( A \Rightarrow B \) to denote \( (A, B) \in R \).

(a) if \( B \subseteq A \), then \( A \Rightarrow B \), \hspace{10mm} (reflexivity)
(b) if \( A \Rightarrow B \), then \( A \cup C \Rightarrow B \cup C \) \hspace{10mm} (augmentation)
(c) if \( A \Rightarrow B \) and \( B \Rightarrow C \), then \( A \Rightarrow C \). \hspace{10mm} (transitivity)

(This set of conditions is called Armstrong’s laws in relational database theory.)

This set of axioms is equivalent to the triple of (a), (b’) and (c) such that

(b’) if \( A \Rightarrow B \) and \( A \Rightarrow C \), then \( A \Rightarrow B \cup C \)

It is also equivalent to the triple of (a), (b’) and (c’) where

(c’) if \( A \Rightarrow B \) and \( B \cup C \Rightarrow D \), then \( A \cup C \Rightarrow D \). \hspace{10mm} (overlap)
An Implicational overlap (c") below can be also deduced from (a), (b), and (c).

(c") if $A \Rightarrow BO$ and $CO \Rightarrow D$, then $AC \Rightarrow D$.

where $B \cap C = \emptyset$, $O \neq \emptyset$.

EXERCISE 17.8 Check the equivalency of these axiom sets $\{a, b, c\}$, $\{a, b', c\}$, and $\{a, b', c'\}$.

17.3 Implicational Systems

Implicational systems have ever been investigated extensively in relational database theory under the name of functional dependencies [119].

Implicational system は Moore family もしくは propositional Horn theory を公理化したものである。Horn presentation と implicational system とは Moore family への対応によって対応づけられている。Horn theories はそのモデルが intersection-closed になるような theories であることが知られている。

We study implicational systems as closure systems. Then the set of rooted circuits provides an implicational base of an implicational system.

Let $E$ be a non-empty finite set. An implicational system $S$ is a collection of ordered pair $(A, B) \in 2^E \times 2^E$ of subsets. When $(A, B) \in S$, we write $A \rightarrow B \in S$ or simply $A \rightarrow B$, and call it an implication. For an implication $A \rightarrow B \in S$, $A$ is called the premise and $B$ is called the conclusion.

If an implicational system $S$ is a inference structure, we call it a full implicational system. Recall the definition of an inference structure:

1. If $A \supseteq B$, then $A \rightarrow B \in S$.
2. If $A \rightarrow B \in S$ and $C \rightarrow D \in S$, then $A \cup C \rightarrow B \cup D \in S$.
3. If $A \rightarrow B \in S$ and $B \rightarrow C \in S$, then $A \rightarrow C \in S$.

This is equivalent to the following.

1. If $A \supseteq B$, then $A \rightarrow B \in S$.
2. If $A \rightarrow B \in S$ and $C \rightarrow D \in S$, then $A \cup C \rightarrow B \cup D \in S$.
3. If $A \rightarrow B \in S$ and $B \cup D \rightarrow C \in S$, then $A \cup D \rightarrow C \in S$.

[注] formal concept analysis では full implicational system は、closed family of implications と呼ばれる。knowledge space や logic では、entail relation と呼ばれる。

relational database では implicational system は、a family of functional dependencies と呼ばれ、full implicational system は、a full family of functional dependencies と呼ばれる。

full implicational system の特徴付けはいくつかあるが、次の条件が使える。

任意の $A' \supseteq A$, $B' \subseteq B$ に対して、$A \rightarrow B$ ならば $A' \rightarrow B'$

A set $X \subseteq E$ is said to respect $A \rightarrow B$ if $A \subseteq X$ implies $B \subseteq X$, or equivalently if either $A \not\subseteq X$ or $B \subseteq X$.

Lemma 17.9 If $X, Y \subseteq E$ respect $A \rightarrow B$, then $X \cap Y$ respects $A \rightarrow B$. 143
(Proof) By assumption, either $A \not\subseteq X$ or $B \subseteq X$ holds. If $A \not\subseteq X$, then clearly $A \not\subseteq X \cap Y$. In case that $A \not\subseteq Y$, the proof is similar. The only left case is that $B \subseteq X$ and $B \subseteq Y$. In this case $B \subseteq X \cap Y$. This completes the proof. □

The following theorem instantly follows from Lemma 17.9.

**Theorem 17.10** The collection, denoted by $\mathcal{K}(S)$, of all the subsets which respect all the implication of an implicational system is a closure system.

Then $S$ is said to be a generating set of $\mathcal{K}(S)$. A minimal generating set of a closure system is called a base. If two implicational systems give rise to the same closure system, they are equivalent.

**Proposition 17.11** If the premise of every implication in an implicational system $S$ is a singleton, the closure system $\mathcal{K}(S)$ forms a distributive lattice.

(Proof) For any $X, Y \in \mathcal{K}(S)$, we shall show $X \cup Y \in \mathcal{K}(S)$. Let $S = \{\{e_j\}, B_j : j = 1, \ldots, m\}$. Suppose $X \cup Y$ does not respect one of the implications, say $\{e_k\} \rightarrow B_k$. That is, we assume that $e_k \in X \cup Y$ and $B_k \not\subseteq X \cup Y$. Then either $e_k \in X$ or $e_k \in Y$ holds. If $e_k \in X$, $B_k \subseteq X$ must hold from the definition of $\mathcal{K}(S)$. Also if $e_k \in Y$, $B_k \subseteq Y$ must hold. In any case, we have $B_k \subseteq X \cup Y$. This is a contradiction. Hence $\mathcal{K}(S)$ is closed under union and intersection. Hence by Proposition 4.3, it is a distributive lattice. □

**Example 17.12** Take an implicational system $S = \{\{1, 2\} \rightarrow \{3, 4\}, \{3\} \rightarrow \{1\}\}$. The implicational lattice induced from $S$ is shown in Fig.17.1.

Suppose that $S$ is an implicational base of a closure system $\mathcal{K}$ and $\tau$ is the corresponding closure operator of $\mathcal{K}$. For a single implication $F \rightarrow G$, we set $K(F \rightarrow G) = \{X \subseteq E : X respects (F \rightarrow G)\}$. Clearly $K(F \rightarrow G)$ is the minimum closure system consisting of all the sets respecting $F \rightarrow G$. Obviously the collection of all the closure systems on a finite fixed set is itself a closure system, and especially a lattice.

**Lemma 17.13** Let $S = \{A_1 \rightarrow B_1, \ldots, A_k \rightarrow B_k\}$ be any implicational system, and suppose $\mathcal{K}(S)$ and $\tau_S$ be the associated closure system and the closure operator, respectively.
(1) \( K(S) = K(A_1 \rightarrow B_1) \land \cdots \land K(A_k \rightarrow B_k) \), and \( X \) respects every \( A \rightarrow B \in S \) if and only if \( B \subseteq \tau_S(A) \).

(2) \( \{ A \rightarrow B : B \subseteq \tau_S(A) \} \) is the minimum inference system containing \( \{ A \rightarrow B : A \rightarrow B \in S \} \).

(Proof) 自分で証明を書く。 

We shall define as follows.

\[
A' = A \cup \bigcup \{ Y : X \subseteq A, X \rightarrow Y \in S \} \quad (17.2)
\]

\[
\overline{A} = A' \cup A'' \cup A''' \cup \cdots \quad (17.3)
\]

Then it is known [158] that

\[
\overline{A} = \tau(A) \quad (17.4)
\]

\( S \) is nonredundant if for any \( X \rightarrow Y \in S \), に対して \( S' = S \setminus \{ X \rightarrow Y \} \) is not equivalent to \( S \) any more.

\( S \) is minimum if \( |S| \leq |S'| \) for any \( S' \) which is equivalent to \( S \). および \( S \) optimal if \( s(S) \leq s(S') \) for any \( S' \) equivalent to \( S \) where \( s(S) = \sum_{X \rightarrow Y \in S} (|X| + |Y|) \).

For any \( A \subseteq E \), we put

\[
A^0 = A \cup \bigcup \{ \tau(X) : X \subseteq A, \tau(X) \subseteq \tau(A) \} \quad (17.5)
\]

\[
A^* = A^0 \cup A^{00} \cup A^{000} \cup \cdots \quad (17.6)
\]

Then \( A \rightarrow A^* \) forms a closure operator, and \( A^* \subseteq \tau(A) \) [158].

\( W \subseteq E \) is quasiclosed if \( W = W^* \), \( W \neq \tau(W) \). A quasiclosed set \( W \) is pseudoclosed if \( W \) is a minimal quasiclosed set, i.e. minimal among the quasiclosed sets \( W' \) with \( \tau(W) = \tau(W') \).

**Theorem 17.14** (Guigues and Duqueness [85], Shock [148], Wild [158]) Let \( K \) be a closure system on \( E \), and \( \tau \) the closure operator.

(1) Suppose \( S \) to be a nonredundant base of \( K \). If every implication of \( S \) is of the form \( X \rightarrow \tau(X) \), \( S \) is a minimum base.

(2) \( S_P = \{ P \rightarrow (\tau(P) \setminus P) : P \text{ is pseudoclosed} \} \) is a canonical minimum base. It is canonical in the sense that for any base \( S' \) of \( K \) and any implication \( P \rightarrow (\tau(P) \setminus P) \) in \( S_P \), \( S' \) contains an implication \( X_P \rightarrow Y_P \) with \( X_P \subseteq P \) and \( \tau(X_P) = P \).

(3) An optimal base is always minimum, and the cardinality \( X_P \) is uniquely determined as \( c_P = \min \{|X| : X \subseteq P, \tau(X) = \tau(P)\} = \min \{|X| : X \subseteq P, X^* = P\} \).
17.4 Minimum bases of implicational systems

The rooted circuits of a closure system $\mathcal{K}$ provides an implicational system $S(\mathcal{C}(\mathcal{K})) = \{X \rightarrow r : (X, r) \in \mathcal{C}(\mathcal{K})\}$. We write simply $X \rightarrow r$ to denote $X \rightarrow \{r\}$.

**Lemma 17.15** For a closure system $(\mathcal{K}, E)$, A set $A \subseteq E$ is a closed set in $\mathcal{K}$ if and only if $A$ respects every implication of $S(\mathcal{C}(\mathcal{K}))$. In particular $S(\mathcal{C}(\mathcal{K}))$ is an implicational system generating $\mathcal{K}$.

(Proof) This is merely a restatement of Proposition 5.15. □

If a rooted circuit $(X, e)$ is a critical circuit, we say that $X$ is a critical stem. As is shown in Proposition 7.13, the collection of critical circuits is necessary and sufficient to determine a convex geometry.

**Proposition 17.16** For a convex geometry $(\mathcal{K}, \tau)$ on $E$, $\{X \rightarrow \tau(X) : X$ is a critical stem\} is a minimum base of $\mathcal{K}$.


**Proposition 17.17** Let $E$ be a finite set in $\mathbb{R}^n$, and $\mathcal{K}$ be the affine convex geometry derived from $E$. Then $S_r = \{X \rightarrow a : (X, a)$ is a critical circuit\} is its optimal base.

(Proof) Suppose $A \subseteq E$ to be a pseudoclosed set. If the convex-hull polytope $P_A$ of $A$ in $\mathbb{R}^n$ is not a simplex, there exists a triangulation of the polytope into multiple simplices. Then $A^* = \tau(A)$, and $A$ is not quasiclosed. Hence $P_A$ must be a simplex. If there is no element of $A$ which being an interior point of $P_A$, then $A^* = \tau(A)$ holds again. Hence there is at least one interior point belonging to $A$. If $P_A$ contains an element other than those in $E \setminus A$, then $A^* = \tau(A)$ holds again. Hence $P_A$ has a unique interior point $a \in A$, and the set of vertices of $P_A$ equals to $A \setminus a$. Then clearly $(A \setminus a, a)$ is a critical circuit. Note that in this case $\sigma(X) = X \cup a$ if $(X, a)$ is critical.

Conversely, as is easy to see, a critical circuit is a pair $(X, e)$ such that $X$ is the vertex set of a simplex and $e$ is its unique interior point. Hence from Theorem 17.14 (2), $S_r$ is a canonical minimum base. Furthermore, $S_r$ satisfies (3) of Theorem 17.14. Hence $S_r$ is an optimal base. □

Wild [158] showed that for a simple binary matroid, $\{C \setminus e \rightarrow e : C$ is a circuit with $\tau(C) = C, e \in C\}$ is its unique optimal implicational base of the closure system of closed sets. In a matroid, a circuit is closed if and only if the stem $C \setminus e$ is critical. Hence the style of the optimal implicational base of a simple binary matroid is equal to that in Proposition 17.17.

17.5 Horn clauses and implicational systems

Let us consider a Boolean algebra on Boolean variables $x_1, \ldots, x_n$. Set $E = \{x_1, \ldots, x_n\}$. For a variable $x$, let $x'$ be the complement of $x$ (i.e. the negation of $x$).

A literal is either a variable or a complement of a variable. $x \land y$ and $x \lor y$ denote the conjunction and the disjunction of literals $x, y$, respectively. A Horn clause is a disjunction of complemented variables and a unique variable. For instance, $x_1 \lor x_2 \lor x_3'$ and $x_2'$ are Horn clauses. A Horn conjunction normal form (Horn CNF) is a conjunction of Horn clauses. For instance, $(x_1 \lor x_2 \lor x_3') \land (x_1 \lor x_1') \land (x_2 \lor x_1')$ is a Horn CNF.
For $A, B \subseteq E$, an implication $A \rightarrow B$ is cryptomorphic to a logical sentence $(a_1 \wedge \cdots \wedge a_k) \Rightarrow (b_1 \wedge \cdots \wedge b_k)$ where $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_m\}$. Actually, a subset $X$ of $E$ gives a truth assignment on the variable. That is, $x_i$ is true if $x_i \in X$ and false if $x_i \not\in X$ ($i = 1, \ldots n$). A sentence $(a_1 \wedge \cdots \wedge a_k) \Rightarrow (b_1 \wedge \cdots \wedge b_m)$ is true under the truth assignment $X$ if and only if $X$ respects an implication $A \rightarrow B$.

An implication is unary if the conclusion is a singleton. A unary implication is cryptomorphic to a Horn clause. That is, for example, a unary implication $\{a_1, \ldots, a_k\} \rightarrow \{b\}$ corresponds to a Horn clause $a_1' \lor \cdots \lor a_k' \lor b$. Hence a Horn CNF is cryptomorphic to a system of unary implications.

A Boolean function is a mapping $f : \{0, 1\}^n \rightarrow \{0, 1\}$ denoted by $f(x_1, \ldots, x_n)$ where $x_1, \ldots, x_n$ are Boolean variables. The following theorem is well known [16].

**Theorem 17.18** A Boolean function of variables $x_1, \ldots, x_n$ can be represented by a Horn CNF $\Gamma$ if and only if the set of truth assignments letting $\Gamma$ true is a closure system.

(Proof) As is seen in Lemma 17.9, the set of truth assignments $X \subseteq E$ which respect the implicational system corresponding to $\Gamma$ is necessarily a closure system.

Conversely, suppose that $K$ is the closure system of the truth assignment letting $\Gamma$ true. Then by Lemma 17.15, an implicational system $S = \{X \rightarrow \{e\} : (X,e) \text{ is a rooted circuit of } K\}$ is a generating set of $K$. Since each implication $X \rightarrow \{e\}$ gives a Horn clause, the logical sentence derived from $S$ is a Horn CNF. $\square$
Chapter 18

Homomorphisms and Convex Functions on Closure Systems

18.1 Blocking pair 上の素朴な max–min lemma

Lemma 18.1 \( L \) を台集合 \( E \) 上の clutter とする。\( b(L) \) をその blocker とする。任意の関数 \( f : E \to \mathbb{R} \) に対して

\[
\max_{X \in L} \min_{x \in X} \{ f(x) \} = \min_{Y \in b(L)} \max_{y \in Y} \{ f(y) \}
\]  

(18.1)

(Proof) まず、左辺 \( \leq \) 右辺 を示す。
任意の \( X \in L, Y \in b(L) \) で \( X \cap Y \neq \emptyset \) だから、ある \( e \in X \cap Y \) があって

\[
\min_{x \in X} \{ f(x) \} \leq f(e) \leq \max_{y \in Y} \{ f(y) \}
\]

\( X, Y \) が任意だから

\[
\max_{X \in L} \min_{x \in X} \{ f(x) \} \leq \min_{Y \in b(L)} \max_{y \in Y} \{ f(y) \}
\]  

(18.2)

次に、上で等号が成立することを示す。
各 \( X \in L \) に対して \( e_x \in X \) を \( f(e_x) = \min_{x \in X} \{ f(x) \} \) をみたす元として、\( M = \{ e_x : X \in L \} \) とおく。
同様に、各 \( Y \in b(L) \) に対して \( e_y \in Y \) を \( f(e_y) = \min_{y \in Y} \{ f(y) \} \) をみたす元として、\( M' = \{ e_y : Y \in b(L) \} \)
とおく。ここで定義から、任意の \( X \in L \) に対して \( X \cap M \neq \emptyset \) で、ゆえに \( M \) は \( L \) の blocker \( b(L) \) の元を含む。つまり、ある \( Y' \in b(L) \) があって、\( Y' \subseteq M \) 同様に、\( M' \) は \( L \) のある元 \( X' \) を含む。定義から
\( X' \cap Y' \neq \emptyset \) だから、\( M \cap M' \neq \emptyset \)。
一方、(18.2) を書き直せば

\[
\max_{e \in M} \{ f(e) \} = \max_{X \in L} \{ f(e_x) \} \leq \min_{Y \in b(L)} \{ f(e_y) \} = \min_{e \in M'} \{ f(e) \}
\]  

(18.3)

ここで、\( M \cap M' \neq \emptyset \) だから必ず等号が成立しなければならない。

18.2 閉集合族集合上の連続写像、凸関数

非空集合 \( E \) がその上の閉集合族・閉集合族対をもつとき、仮に離散位相空間と呼ぶことにする。

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Def. 18.2 離散数位空間 $E$ から $E'$ への写像 $f: E \to E'$ が、閉集合の逆像が常に閉集合になるとき、$f$ は連続であるということにする。これは、閉集合の逆像が常に閉集合であるというのも同じである。

Def. 18.3 各点 $e \in E$ に対して $e$ を含む極小な閉集合 $N$ をその閉近傍と呼ぶことにする。つまり根付きコサーキット $(Y, e) \in D$ に対して、$N = Y \cup e$ となる集合のこと。

Proposition 18.4 写像 $f: E \to E'$ が連続で、$E$ が連結であれば、$f$ の像も（その相対位相に関して）連結になる。

(Proof) 定義から自明。 □

Proposition 18.5 写像 $f: E \to E'$ が連続であるための必要十分条件は、$E'$ の任意の元 $e'$ の開近傍 $N'$ に対して $f^{-1}(N')$ が $E$ の開集合になることである。

(Proof) 定義から、開近傍 $N'$ の全体は、$O(E')$ の join-irreducible 元の全体に一致することから明らか。 □

Def. 18.6 整数全体の集合上での標準の開集合族 $\mathcal{O}$ は、閉集合族 $K$ を以下で定義する。

$$\mathcal{O} = \{ a, a + 1, a + 2, \ldots : a \in \mathbb{Z} \}$$

$\mathbb{Z}$ で整数全体から整数への写像が連続であるとは、上の開集合族に関して連続であると考えることにする。

Def. 18.7 離散数位空間上の整数値関数 $f: E \to \mathbb{Z}$ が凸関数であるとは、各根付きサーキット $(X, e) \in C$ に対して

$$f(e) \leq \min_{x \in X} f(x)$$

をみたすこととする。この条件は以下のように書いても同じ。

$$f(e) \leq \max_{X \in C(e)} \min_{x \in X} f(x) \quad (e \in E)$$

(18.4)

Def. 18.8 整数値関数 $f: E \to \mathbb{Z}$ が凹関数であるとは、

$$f(e) \geq \max_{Y \in D(e)} \min_{y \in Y} f(y)$$

をみたすこととする。

凸かつ凹であれば Lemma 18.1 から、上の (18.5), (18.6) で等号が成立しなければならないとわかる。ゆえにこれか直ちに、集合 $S_c = \bigcup \{ X : X \in C(e) \} = \bigcup \{ Y : Y \in D(e) \}$ 上で $f$ は定数関数であると分かる。離散数位空間の連結成分は、Proposition 5.19 からその隣接グラフの連結成分に一致するから、凸かつ凹であれば連結成分上で定数になる。

結局、これをまとめてと、

Proposition 18.9 離散数位空間上で、凸かつ凹である関数は、各連結成分上で定数関数になる。

Proposition 18.10 凸かつ凹上の関数 $f: E \to \mathbb{Z}$ が凸関数であれば、連続写像になる。
(証明) $f(E) = \{a_1, a_2, \ldots, a_k\}$ で $a_1 > a_2 > \cdots > a_k$ と仮定する。このとき、任意の $j = 1, \ldots, k$ で $A_j = f^{-1}(\{a_1, \ldots, a_j\})$ が開集合であることを示せばよい。それには、任意の根付きサーキット $(X, e) \in C$ に対して $(X \cup e) \cap A_j \neq \{e\}$ であることを示せばよい。(Note: ここで定義から、常に $X \neq \emptyset$ であることが保証されている。)

そうでないとする。つまり、$e \in A_j$ かつ $X \cap A_j = \emptyset$ であったとする。すると定義から、任意の $t \in X$ で $f(t) < a_j$。ゆえに、

$$\max_{t \in X} \{f(t)\} < a_j$$

一方、$e \in A_j$ より $a_j \leq f(e)$ であるが、これは凸関数の定義に矛盾する。□

「課題」「凸関数の局所的な最小値は、実は大域的な最小値になる。」という命題が成立するようにうまく「局所的に最小」を定義したい。どうすればいいか？次の定義は、うまくいか？

Def. 18.11 $f : E \to \mathbb{Z}$ が点 $e \in E$ で極小であるとは、

$$f(e) \leq \min_{t \in S_e} \{f(t)\}$$

(18.7)

18.3 凸幾何の準同型

Theorem 18.12 Let $(A, K)$ be a dual pair of an antimatroid and a convex geometry on a finite set $E$, and $f : E \to E'$ be a surjective map. Then $A' = \{f(A) : A \in A\}$ is an antimatroid on $E'$, and $K' = \{E' - f(E - B) : B \in K\}$ is a convex geometry which is dual to $A'$.
Chapter 19

Proofs of Exercises

19.1 Proofs of Exercises

- (Proof of Exercise 6.6)
  \[ C \cap D = \{e\} \] であったとする。 \( X = C \setminus e, Y = D \setminus e \) とするとそれぞれ \( M \) と \( M^* \) の独立集合になるから、\( E \) の分割 \( E = B \cup B^* \) があって \( X \subseteq B, Y \subseteq B^* \) で、\( e \in B \) と \( e \in B^* \) のどちらであっても矛盾。 □

- (Proof of Exercise 6.13)
  もし \( D = E \setminus H \) が \( M^* \) で独立であると、補題 6.5 から、\( H \) が \( M \) の基を含むことになり、矛盾。ゆえに \( D \) は \( M^* \) で従属集合であり、その極小性は hyperplane の極大性から導かれ、ゆえに \( D \) はコサーキットである。逆も明らか。 □

- (Proof of Exercise 2.1):
  For a family \( H \subseteq 2^E \), \( H \uparrow \) denotes \( \{Y \in 2^E : X \subseteq Y \text{ for some } X \in H.\} \). In order to prove \( b(b(L)) = L \), it is sufficient to show \( b(b(L)) \uparrow = L \uparrow \).

  Suppose \( X \in L \uparrow \). By definition, \( X \) intersects every member of \( b(H) \). Hence \( X \in b(b(L)) \uparrow \).

  Conversely suppose \( X \notin L \uparrow \). If \( E \setminus X \) does not intersect a certain element \( Z \in L, X \) includes \( Z \), a contradiction. Hence \( E \setminus X \) intersects every element of \( L \), and so \( E \setminus X \in b(L) \uparrow \). Hence there is an element \( W \in b(L) \) such that \( W \subseteq E \setminus X \), which readily implies \( X \cap W = \emptyset \) and \( X \notin b(b(L)) \uparrow \). This completes the proof. □

- (Proof of Exercise 17.4):
  Suppose \( f \) is a closure function. Obviously \( f(A) \cup f(B) \subseteq f(A \cup B) \). Then \( f(f(A) \cup f(B)) \subseteq f(f(A) \cup f(B)) = f(A \cup B) \subseteq f(f(A) \cup f(B)) \). Hence \( f(A \cup B) = f(f(A) \cup f(B)) \).

  Suppose conversely \( f \) is path-independent. Then \( f(f(A)) = f(f(A) \cup f(A)) = f(A \cup A) = f(A) \).

  Hence it is idempotent. If \( A \subseteq B \), then \( f(B) = f(A \cup B) = f(f(A) \cup f(B)) \supseteq f(A) \cup f(B) \supseteq f(A) \).

  Thus \( f \) is monotone. □
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