S セメスター 全学体験ゼミナール「じっくり学ぶ数学 I」
レポート問題 (その 9)

問 1. 3 行 3 列の行列

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

を考えることと, $\mathbb{R}^3$ の三つのベクトル

$$\mathbf{a}_1 = \begin{pmatrix} a \\ d \\ g \end{pmatrix},\ \mathbf{a}_2 = \begin{pmatrix} b \\ e \\ h \end{pmatrix},\ \mathbf{a}_3 = \begin{pmatrix} c \\ f \\ i \end{pmatrix} \in \mathbb{R}^3$$

を考えることは同じことであると解釈して, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^3$ を変数とする関数 $f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ について考える. このとき, 次の問に答えよ.

(1) $f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ が,

(イ) 多重線型性: 勝手なベクトル $\mathbf{a}_1, \mathbf{a}'_1, \mathbf{a}_2, \mathbf{a}'_2, \mathbf{a}_3, \mathbf{a}'_3 \in \mathbb{R}^3$ と, 勝手な実数 $c \in \mathbb{R}$ に対して, 次が成り立つ.

$$\begin{cases} f(\mathbf{a}_1 + \mathbf{a}'_1, \mathbf{a}_2, \mathbf{a}_3) = f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) + f(\mathbf{a}'_1, \mathbf{a}_2, \mathbf{a}_3) \\ f(c \cdot \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = c \cdot f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \end{cases}$$

$$\begin{cases} f(\mathbf{a}_1, \mathbf{a}_2 + \mathbf{a}'_2, \mathbf{a}_3) = f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) + f(\mathbf{a}_1, \mathbf{a}'_2, \mathbf{a}_3) \\ f(\mathbf{a}_1, c \cdot \mathbf{a}_2, \mathbf{a}_3) = c \cdot f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \end{cases}$$

$$\begin{cases} f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 + \mathbf{a}'_3) = f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) + f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}'_3) \\ f(\mathbf{a}_1, \mathbf{a}_2, c \cdot \mathbf{a}_3) = c \cdot f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \end{cases}$$

という性質を持つとき, $f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ は,

$$\begin{aligned} f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = & abc \cdot f(\mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_1) + abf \cdot f(\mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_2) + abi \cdot f(\mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_3) \\ & + aec \cdot f(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1) + aef \cdot f(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_2) + aei \cdot f(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \\ & + ahc \cdot f(\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_1) + ahf \cdot f(\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_2) + ahi \cdot f(\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_3) \\ & + dbc \cdot f(\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_1) + dbf \cdot f(\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_2) + dbi \cdot f(\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3) \\ & + dec \cdot f(\mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_1) + def \cdot f(\mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_2) + dei \cdot f(\mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_3) \\ & + dhc \cdot f(\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1) + dhf \cdot f(\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_2) + dhi \cdot f(\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_3) \\ & + gbc \cdot f(\mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_1) + gbf \cdot f(\mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_2) + gbi \cdot f(\mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_3) \\ & + gec \cdot f(\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1) + gef \cdot f(\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_2) + gei \cdot f(\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_3) \end{aligned}$$

$$+ ghc \cdot f(\mathbf{e}_3, \mathbf{e}_3, \mathbf{e}_1) + ghf \cdot f(\mathbf{e}_3, \mathbf{e}_3, \mathbf{e}_2) + ghi \cdot f(\mathbf{e}_3, \mathbf{e}_3, \mathbf{e}_3)$$

というように二十七個の項の和の形に表わせることを示せ. ただし,

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^3$$

と表わした.

(2) $f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ が, (イ) の「多重線型性」という性質とともに,

(ロ) 歪対称性：勝手なベクトル $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^3$ に対して, 次が成り立つ.

$$\begin{aligned} f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) &= -f(\mathbf{a}_2, \mathbf{a}_1, \mathbf{a}_3) && (\ \mathbf{a}_1 \leftrightarrow \mathbf{a}_2\ ) \\ f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) &= -f(\mathbf{a}_3, \mathbf{a}_2, \mathbf{a}_1) && (\ \mathbf{a}_1 \leftrightarrow \mathbf{a}_3\ ) \\ f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) &= -f(\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_2) && (\ \mathbf{a}_2 \leftrightarrow \mathbf{a}_3\ ) \end{aligned}$$

という性質も持つとき, $f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ は,

$$\begin{aligned} f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) =& aei \cdot f(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) + ahf \cdot f(\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_2) + dbi \cdot f(\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3) \\ & + dhc \cdot f(\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1) + gbf \cdot f(\mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_2) + gec \cdot f(\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1) \\ =& (aei - ahf - dbi + dhc + gbf - gec) \cdot f(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \end{aligned}$$

と表わせることを示せ.

(3) $f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ が, (イ) の「多重線型性」, (ロ) の「歪対称性」という二つの性質とともに,

(ハ) 規格化条件：$f(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = 1$

という性質も持つとき,

$$f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = aei - ahf - dbi + dhc + gbf - gec$$

と表わせることを示せ.

問 2. 4 行 4 列の行列

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

を考えることと, $\mathbb{R}^4$ の四つのベクトル

$$\mathbf{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{pmatrix},\ \mathbf{a}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{pmatrix},\ \mathbf{a}_3 = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{pmatrix},\ \mathbf{a}_4 = \begin{pmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{pmatrix} \in \mathbb{R}^4$$

を考えることは同じことであると解釈して, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4 \in \mathbb{R}^4$ を変数とする関数 $f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$ について考える. このとき, 次の問に答えよ.

(1) $f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$ が,「多重線型性」,「歪対称性」という二つの性質を持つとき, $f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$ は,

$$\begin{aligned}
&f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) \\
&\quad = a_{11}a_{22}a_{33}a_{44} \cdot f(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4) + a_{11}a_{22}a_{43}a_{34} \cdot f(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_3) \\
&\qquad + a_{11}a_{32}a_{23}a_{44} \cdot f(\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_4) + a_{11}a_{32}a_{43}a_{24} \cdot f(\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_2) \\
&\qquad + a_{11}a_{42}a_{23}a_{34} \cdot f(\mathbf{e}_1, \mathbf{e}_4, \mathbf{e}_2, \mathbf{e}_3) + a_{11}a_{42}a_{33}a_{24} \cdot f(\mathbf{e}_1, \mathbf{e}_4, \mathbf{e}_3, \mathbf{e}_2) \\
&\qquad + a_{21}a_{12}a_{33}a_{44} \cdot f(\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4) + a_{21}a_{12}a_{43}a_{34} \cdot f(\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_4, \mathbf{e}_3) \\
&\qquad + a_{21}a_{32}a_{13}a_{44} \cdot f(\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_4) + a_{21}a_{32}a_{43}a_{14} \cdot f(\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_1) \\
&\qquad + a_{21}a_{42}a_{13}a_{34} \cdot f(\mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_1, \mathbf{e}_3) + a_{21}a_{42}a_{33}a_{14} \cdot f(\mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_3, \mathbf{e}_1) \\
&\qquad + a_{31}a_{12}a_{23}a_{44} \cdot f(\mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4) + a_{31}a_{12}a_{43}a_{24} \cdot f(\mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_4, \mathbf{e}_2) \\
&\qquad + a_{31}a_{22}a_{13}a_{44} \cdot f(\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_4) + a_{31}a_{22}a_{43}a_{14} \cdot f(\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_1) \\
&\qquad + a_{31}a_{42}a_{13}a_{24} \cdot f(\mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_1, \mathbf{e}_2) + a_{31}a_{42}a_{23}a_{14} \cdot f(\mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_2, \mathbf{e}_1) \\
&\qquad + a_{41}a_{12}a_{23}a_{34} \cdot f(\mathbf{e}_4, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) + a_{41}a_{12}a_{33}a_{24} \cdot f(\mathbf{e}_4, \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_2) \\
&\qquad + a_{41}a_{22}a_{13}a_{34} \cdot f(\mathbf{e}_4, \mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3) + a_{41}a_{22}a_{33}a_{14} \cdot f(\mathbf{e}_4, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1) \\
&\qquad + a_{41}a_{32}a_{13}a_{24} \cdot f(\mathbf{e}_4, \mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_2) + a_{41}a_{32}a_{23}a_{14} \cdot f(\mathbf{e}_4, \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1) \\
&\quad = (a_{11}a_{22}a_{33}a_{44} - a_{11}a_{22}a_{43}a_{34} - a_{11}a_{32}a_{23}a_{44} \\
&\qquad + a_{11}a_{32}a_{43}a_{24} + a_{11}a_{42}a_{23}a_{34} - a_{11}a_{42}a_{33}a_{24} \\
&\qquad - a_{21}a_{12}a_{33}a_{44} + a_{21}a_{12}a_{43}a_{34} + a_{21}a_{32}a_{13}a_{44} \\
&\qquad - a_{21}a_{32}a_{43}a_{14} - a_{21}a_{42}a_{13}a_{34} + a_{21}a_{42}a_{33}a_{14} \\
&\qquad + a_{31}a_{12}a_{23}a_{44} - a_{31}a_{12}a_{43}a_{24} - a_{31}a_{22}a_{13}a_{44} \\
&\qquad + a_{31}a_{22}a_{43}a_{14} + a_{31}a_{42}a_{13}a_{24} - a_{31}a_{42}a_{23}a_{14}
\end{aligned}$$

$$
\begin{aligned}
&- a_{41}a_{12}a_{23}a_{34} + a_{41}a_{12}a_{33}a_{24} + a_{41}a_{22}a_{13}a_{34} \\
&-a_{41}a_{22}a_{33}a_{14} - a_{41}a_{32}a_{13}a_{24} + a_{41}a_{32}a_{23}a_{14}) \cdot f(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)
\end{aligned}
$$

というように二十四個の項の和の形に表わせることを示せ. ただし,

$$
\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},\ \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ \mathbf{e}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^4
$$

と表わした.

(2) $\{1, 2, 3, 4\}$ という四文字の置換全体の集合を,

$$
\mathfrak{S}_4 = \left\{ \sigma : \{1,2,3,4\} \to \{1,2,3,4\} \;\middle|\; \begin{array}{l} \{\sigma(1), \sigma(2), \sigma(3), \sigma(4)\} \text{ は} \\ \{1,2,3,4\} \text{ の入れ換え} \end{array} \right\}
$$

と表わす. このとき, (1) の結果は,

$$
\begin{aligned}
f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) &= \sum_{\sigma \in \mathfrak{S}_4} a_{\sigma(1)1} a_{\sigma(2)2} a_{\sigma(3)3} a_{\sigma(4)4} \cdot f(\mathbf{e}_{\sigma(1)}, \mathbf{e}_{\sigma(2)}, \mathbf{e}_{\sigma(3)}, \mathbf{e}_{\sigma(4)}) \\
&= \left(\sum_{\sigma \in \mathfrak{S}_4} \operatorname{sgn} \sigma \cdot a_{\sigma(1)1} a_{\sigma(2)2} a_{\sigma(3)3} a_{\sigma(4)4} \right) \cdot f(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)
\end{aligned}
$$

と表わせることを示せ. ただし,

$$
f(\mathbf{e}_{\sigma(1)}, \mathbf{e}_{\sigma(2)}, \mathbf{e}_{\sigma(3)}, \mathbf{e}_{\sigma(4)}) = \operatorname{sgn} \sigma \cdot f(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)
$$

と表わした. ($\operatorname{sgn} \sigma = \pm 1$ を「置換 σ の符号」と言う.)